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JUN 21 1937

NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XI BATON ROUGE, LA., MAY, 1937 No. 8

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Problem Department

Reviews and Abstracts

PUBLISHED BY LOUISIANA STATE UNIVERSITY

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Published by the Louisiana State University Press

Club Rates May Be Had on Application

Subscription, $1.50 Per Year in Advance Single Copies, 20c.

Vol. XI BATON ROUGE, LA. MAY, 1937 No. 8

Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

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1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Quality and Mathematics

A review published by William L. Schaaf in a recent issue of *Scripta Mathematica* quotes the following words from Charles W. R. Hooker's "*What Is the Fourth Dimension*?": "The very perfection of mathematics is a snare, as it serves to draw attention from what it cannot do." In the same review is attributed to Hooker the confession that a "very real weakness of mathematics is that it deals only with quantity and cannot, frankly, deal with quality—the very things that probably count for most, those not measurable as numbers at all."

It smacks of dogmatism to say what mathematics cannot do. Tradition has it that a certain scientist once went up and down the country proving to audiences that flying machines are impossible. Today aeroplanes freely sail the heavens, and, with the aid of mathematics the impossible has been accomplished.

Students of the physical sciences, using only the applied phases of mathematics are prone to regard it as a purely quantitative science. Few of them fully appreciate that the most sweeping advances in mathematics began only after certain mathematicians broke through the trammels confining the science to quantity measurement. Then came such mathematical conceptions as Order, Form, Group, Matrix, Class, all logically independent of mere number, and each the key to a powerful and growing branch of the science.

To assert that the *quality* of a thing is not amenable to mathematical process is to assert the existence of domains of reality of whose nature and properties a knowledge may be acquired *without* the aids of precise definition, logical deduction, or clarified and classified concepts. All of these are the very inner essence of mathematics, properly conceived. Knowledge without definiteness is no knowledge at all.

The term *quality* would not have the half-dozen meanings assigned to it today by the lexicographers if it had been mathematically treated as long ago was the term quantity. Quantity is a definite answer to How much, or, How many. Quality is a definite answer to nothing.

Not until an entity, or a state of being of an entity, or a quality of an entity has been identified by *defining* characteristics, can it be made a subject of proper thought. Once it has been identified in such manner, mathematics, and mathematics alone, is the carrier ordained to bring the answers to questions without which *all notions* of quality must be vain and useless. If quality notions are to be useful forms of knowledge, qualities must be *compared* and the degree of a quality must be *measured*; and, where *measure* and *comparison* are *there* must mathematics be found.

S. T. SANDERS.

A Generalization of Trigonometry

By H. P. THIELMAN
College of St. Thomas

Most college students of mathematics are familiar with the idea that there exist several kinds of geometries and algebras. It has not occurred to many that there are various types of trigonometries. We give here a generalized trigonometry, special cases of which are familiar to every college student of mathematics.

Let us define an operation "$\cdot$" which will be assumed to possess the following properties of ordinary multiplication: It is associative, cummutative, and distributive with respect to ordinary addition. This operation, which will be called multiplication, will be applied to an arbitrary class of single valued functions. Repeated application of this operation to the same function will be indicated by positive integral exponents. Thus $F(x)\cdot F(x)$ will be written as $F(x)^2$, $F(x)\cdot F(x)\cdot F(x)$ as $F(x)^3$, and so on. The symbol for ordinary multiplication will be omitted, namely k multiplied by $F(x)$ in the ordinary sense will be written $kF(x)$. We shall add the further hypothesis that $F(x)\cdot aG(x)=aF(x)\cdot G(x)$ if a is a constant.

One of the functions under consideration, say $f(x)$, will be assumed to have the properties

$$f(x)\cdot f(y)=f(x+y). \qquad f(0)\neq 0. \tag{1}$$

Next we define two functions $s(x)$ and $c(x)$ by the equations

$$s(x)=\frac{f(x)-f(-x)}{2k},\quad c(x)=\frac{f(x)+f(-x)}{2} \tag{2}$$

where $k\neq 0$ is a constant. We see that $s(0)=0$, $c(0)=f(0)$, $s(-x)=-s(x)$, and $c(-x)=c(x)$. Multiplying (in the ordinary sense) the first equation of (2) first by k, and then by $-k$, and adding each of the results to the second one of those equations, we obtain

$$\begin{aligned} c(x)+ks(x)&=f(x),\\ c(x)-ks(x)&=f(-x). \end{aligned} \tag{3}$$

Multiplying these last two equations we get, because of (1),

$$c(x)^2-[ks(x)]^2=f(0). \tag{4}$$

Next we can state a generalized De Moivre's theorem

$$[c(x)+ks(x)]^n=f(x)^n=f(nx)=c(nx)+ks(nx), \tag{5}$$

which follows at once from (1) and (3). Also by (3) and (1) we have

$$\begin{aligned} c(x+y)+ks(x+y)&=[c(x)+ks(x)]\cdot[c(y)+ks(y)]\\ &=c(x)\cdot c(y)+c(x)\cdot ks(y)+c(y)\cdot ks(x)+ks(x)\cdot ks(y),\end{aligned}$$

$$\begin{aligned} c(x+y)-ks(x+y)&=[c(x)-ks(x)]\cdot[c(y)-ks(y)]\\ &c(x)\cdot c(y)-c(x)\cdot ks(y)-c(y)\cdot ks(x)+ks(x)\cdot ks(y).\end{aligned}$$

Adding and subtracting the last two equations we get the addition formulas

$$c(x+y)=c(x)\cdot c(y)+ks(x)\cdot kas(y), \tag{6}$$

$$s(x+y)=s(x)\cdot c(y)+s(y)\cdot c(x).$$

If we assume $f(x)$ to be differentiable we get from (1)

$$f'(x+y)=f'(x)\cdot f(y).$$

Let $x=0$, then

$$f'(y)=f'(0)\cdot f(y). \tag{7}$$

Hence to differentiate $f(x)$ with respect to x is equivalent to multiplying it by $f'(0)$. Therefore,

$$s'(x)=\frac{f'(0)}{k}\cdot\frac{f(x)+f(-x)}{2}=\frac{f'(0)}{k}\cdot c(x), \tag{8}$$

$$c'(x)=kf'(0)\cdot\frac{f(x)-f(-x)}{2k}=kf'(0)\cdot s(x).$$

If we let $f(x)=e^{ix}$, $k=ni$, where n is a real number, and $i^2=-1$, and if we let "$\cdot$" stand for ordinary multiplication, equation (4) becomes

$$c(x)^2+n^2s(x)^2=1. \tag{9}$$

Let $c(x)=X$, and $s(x)=Y$, where X and Y are to be interpreted as rectangular coordinates in the plane. By (9) X and Y are seen to lie on the ellipse

$$X^2+n^2Y^2=1,$$

and hence, in this case, we obtain elliptical trigonometries. For $n=1$ we get the ordinay circular trigonometry.

If $f(x)=e^x$, and $k=1$, we get the theory of the hyperbolic functions sinh x, and cosh x.

A somewhat stranger type of trigonometry is obtained by letting $f(x)$ be a matrix $A(x)$, which has the properties (1) under matrix multiplication and addition. Then $s(x)$ and $c(x)$ will also be matrices.

Other examples of trigonometries could be given in the so called theory of functions of composition, a particular case of which would include the theory of the Bessel functions ber x and bei x. To obtain this trigonometry of Bessel functions, we let $f(a)=e^{ia1*}$, and $k=i$.†

A trigonometry of hypergeometric series can be obtained by making use of the theory of functions of composition in $2n$ variables, where n is a positive integer.†

A further extension of the original generalization can be obtained by starting out with functions of several variables. We then assume the existence of a function $f(x_1, x_2, \cdots, x_n)$ which has the properties

$$f(x_1, x_2, \cdots, x_n) \cdot f(y_1, y_2, \cdots, y_n) = f(x_1+y_1, x_2+x_2, \cdots, x_n+y_n),$$

$$f(0, 0, \cdots, 0) \neq 0,$$

and define $s(x_1, x_2, \cdots, x_n)$ and $c(x_1, x_2, \cdots, x_n)$ in analogy with $s(x)$ and $c(x)$ in equation (2). A particular case of a trigonometry of this kind is obtained by letting $f(x_1, x_2, \cdots, x_n)$ be equal to $a_1^{x_1} a_2^{x_2} \cdots a_n^{x_n}$, where $a_1, a_2, \cdots, a_n$ are positive numbers, which might be called the bases of this trigonometry.

†For the meaning of e^{ia1*} see Vito Volterra, *Theory of Functionals and of Integral and Integro-Differential Equations*, translated by Miss M. Long, p. 104, or the author's paper, *Annals of Mathematics*, Sec. Series, Vol. 31, p. 198.

‡For the definition of the composition of functions in $2n$ variables see the author's paper to which the first footnote refers, p. 202.

On the Pell Equation

By HOWARD D. GROSSMAN
De Witt Clinton High School, Bronx, N. Y.

Among Diophantine Equations the Pell Equation $x^2-dy^2=1$ ($0<d\neq\square$, all letters representing integers) has always had a strong esthetic appeal for mathematicians by reason of both the simplicity of its form and the elegance of its solution. It is not surprising therefore that this equation has had a long history and that many relations between its solutions have been discovered. If $x^2-dy^2=-1$ has a solution in positive integers and (t_1, u_1) is the smallest one, then the nth solution, in order of increasing magnitude, of $x^2-dy^2=\pm 1$ is (t_n, u_n) where $t_n+u_n\sqrt{d}=(t_1+u_1\sqrt{d})^n$ and $t^2_n-du^2_n=(-1)^n$. This is a simple extension of Carmichael's *Diophantine Analysis*, p. 31. The heart of the proof is that, since also $t_n-u_n\sqrt{d}=(t_1-u_1\sqrt{d})^n$, then

$$t_n^2-du_n^2=(t_1^2-du_1^2)^n=(-1)^n.$$

Theorem: *If $x^2-dy^2=-1$ has a solution, then*

$$t_{n+k}=2t_kt_n-(-1)^kt_{n-k}$$

and

$$u_{n+k}=2t_ku_n-(-1)^ku_{n-k}.$$

Proof: Since

$$t_r+u_r\sqrt{d}=(t_1+u_1\sqrt{d})^r,*$$

$$t_{n+k}+u_{n+k}\sqrt{d}=(t_k+u_k\sqrt{d})^2(t_{n-k}+u_{n-k}\sqrt{d}).$$

$$\therefore t_{n+k}=t_k^2t_{n-k}+2dt_ku_ku_{n-k}+dt_{n-k}u_k^2$$

$$=2t_k(t_kt_{n-k}+du_ku_{n-k})-t_{n-k}(t_k^2-du_k^2)$$

and

$$u_{n+k}=t_k^2u_{n-k}+2t_kt_{n-k}u_k+du_k^2u_{n-k}$$

$$=2t_k(t_ku_{n-k}+t_{n-k}u_k)-u_{n-k}(t_k^2-du_k^2).$$

But

$$t_n+u_n\sqrt{d}=(t_k+u_k\sqrt{d})(t_{n-k}+u_{n-k}\sqrt{d})$$

$$=(t_kt_{n-k}+du_ku_{n-k})+(t_ku_{n-k}+t_{n-k}u_k)\sqrt{d}$$

*In this formula r is usually taken a positive integer but it may be taken 0 (yielding $t_0=1$, $u_0=0$) or a negative integer (yielding $t_{-r}=(-1)^rt_r$, $u_{-r}=-(-1)^ru_r$, since $t_{-r}+u_{-r}\sqrt{d})=(t_1+u_1\sqrt{d})^{-r}=(t_1^2-du_1^2)^{-r}(t_1-u_1\sqrt{d})^r=(-1)^{-r}(r_r-u_r\sqrt{d})$. The extension to fractional subscripts suggests itself as an interesting exercise for the reader.

$$\therefore t_{n+k}=2t_k t_n-(-1)^k t_{n-k}$$

and $$u_{n+k}=2t_k u_n-(-1)^k u_{n-k}.$$

Whether $x^2-dy^2=-1$ has or has not solutions, $x^2-dy^2=1$ always has (Carmichael's *Diophantine Analysis*, p. 31).

Theorem: *If* (t_n, u_n) *is, in order of increasing magnitude, the nth solution in positive integers of* $x^2-dy^2=1$, *then*

$$t_{n+k}=2t_k t_n-t_{n-k}$$

and $$u_{n+k}=2t_k u_n-u_{n-k}.$$

(See Dickson's *History of the Theory of Numbers*, volume 2, pages 384, 386, 397.)

Proof: Same as before except that $t_k^2-du_k^2=1$.†

†Here $t_{-r}=t_r$, $u_{-r}=-u_r$.

These recursion formulas offer an easier method of computing successive solutions of $x^2-dy^2=\pm 1$ or of raising $t_1+u_1\sqrt{d}$ to successive positive integral powers.

E. g., for $x^2-2y^2=\pm 1$, $t_{n+1}=2t_n+t_{n-1}$, $u_{n+1}=2u_n+u_{n-1}$

$t=1, 3, 7, 17, 41, 99, 239, 577, \cdots,$

$u=1, 2, 5, 12, 29, 70, 169, 408, \cdots;$

for $x^2-3y^2=1$, $t_{n+1}=4t_n-t_{n-1}$, $u_{n+1}=4u_n-u_{n-1}$

$t=2, 7, 26, 97, 362, 1351, \cdots,$

$u=1, 4, 15, 56, 209, 780, \cdots.$

These tables begin at (t_1, u_1) but they may be extended backwards as well as forwards infinitely.

J

Corrigenda

From the last two lines of p. 296, Vol. XI, delete the sentence "Figures 1, 2 and 3 illustrate the three cases."

In line 12, p. 334, Vol. XI, the word "numerator" should read "denominator."

In the same line the expression $\sqrt{\dfrac{x(x+a)}{x}}$ should be replaced by $\dfrac{\sqrt{x(x+a)}}{x}$

In line 22, p. 346, Vol. XI, the word "Whitman" should read "Whitworth."

N Producers in Cooperation and Competition*

By A. A. AUCOIN
Louisiana State University

1. *Introduction.* In the theory of mathematical economics n producers may align themselves in one of several manners. First, if they wish, as a group, to make a maximum profit, they are said to be in *Cooperation.* Secondly, if each producer assumes that his own production is independent of the others and tries to make his profit a maximum, they are said to be operating under *Cournot Competition.* Finally, if the price is regarded as fixed, and each producer attempts to make his profit a maximum, the producers are said to be operating under *Strict Competition.* We consider here, these three problems.

We shall assume in all three of the above cases that the cost functions are given by

$$q_i(u_i) = A_i u_i^2 + B_i u_i + C_i, \qquad (i = 1, 2, \ldots, n)$$

where u_i is the number of units produced and $q_i(u_i)$ is the cost of production of the ith producer, and A_i, B_i, C_i are positive constants.

If y is the demand and p the unit price, we assume that

$$y = ap + b, \qquad a < 0,\ b > 0$$

Further, the supply is assumed equal to the demand, i. e.,

$$y = \sum_{k=1}^{n} u_k = ap + b.$$

The profit for the ith producer will be

$$\pi_i = u_i p - q_i(u_i),$$

while the total profit is given by

$$\pi = \sum_{i=1}^{n} \pi_i = \sum_{i=1}^{n} (u_i p - q_i(u_i)).$$

2. *Cooperation.* If the producers are operating in Cooperation we wish to determine the amount each should produce, and at what

*This is a generalization of Sections 16, 17, 18, pp. 23-27 of *Mathematical Introduction to Economics* by G. C. Evans, McGraw-Hill Book Co., New York, (1930).

price he should sell, in order that the total profit, π, shall be a maximum. Since

$$\frac{\partial \pi}{\partial u_i} = -\frac{1}{a}\sum_{k=1}^{n} u_k - \frac{b}{a} + \frac{1}{a}\sum_{k=1}^{n} u_k - 2A_i u_i - B_i, \qquad (i=1,2,\cdots,n)$$

it is necessary† that $u_1, u_2, \cdots, u_n$ satisfy the n equations

$$\begin{aligned}
2u_1(1-aA_1) + 2u_2 + \cdots\cdots + 2u_n &= b + aB_1, \\
2u_1 + 2u_2(1-aA_2) + \cdots\cdots + 2u_n &= b + aB_2, \\
\cdots\cdots\cdots

Since

$$\Delta_2 = \begin{vmatrix} T_1 & 1 \\ B_1 - B_2 & A_2 \end{vmatrix} = P_2 \left(\frac{T_1}{A_1} - \sum_{j=1}^{2} \frac{B_1 - B_j}{A_1 A_j} \right)$$

we assume that

$$\Delta_k = P_k \left(\frac{T_1}{A_1} - \sum_{j=1}^{k} \frac{B_1 - B_j}{A_1 A_j} \right).$$

By the recursion formula

$$\Delta_{k+1} = A_{k+1}\Delta_k - (B_1 - B_{k+1}) A_2 A_3 \cdots A_k$$

$$= A_{k+1} P_k \left(\frac{T_1}{A_1} - \sum_{j=1}^{k} \frac{B_1 - B_j}{A_1 A_j} \right) - \frac{B_1 - B_{k+1}}{A_1 A_{k+1}} P_{k+1}$$

$$= P_{k+1} \left(\frac{T_1}{A_1} - \sum_{j=1}^{k+1} \frac{B_1 - B_j}{A_1 A_j} \right)$$

which completes the induction.

We may write

$$D = 2(2a)^{n-1} \Delta'_n$$

where

$$\Delta'_n = \begin{vmatrix} 1 - aA_1 & A_1 & A_1 \cdots\cdots A_1 \\ 1 & -A_2 & 0 \cdots\cdots 0 \\ 1 & 0 & -A_3 \cdots\cdots 0 \\ \cdots & \cdots & \cdots \\ 1 & 0 & 0 \cdots\cdots -A_n \end{vmatrix}$$

and satisfies the recursion formula

$$\Delta'_n = -A_n \Delta'_{n-1} - (-1)^{n-2} A_1 A_2 \cdots A_{n-1},$$

where Δ'_{n-1} is the principal minor of order $n-1$ in the upper left hand corner of Δ'_n.

It follows by induction that

$$\Delta'_n = (-1)^{n-1} P_n (S_n - a)$$

where

$$S_n = \sum_{k=1}^{n} \frac{1}{A_k}.$$

We have then that

$$u_1 = \frac{(-2a)^{n-1}P_n\left(\dfrac{T_1}{A_1} - \displaystyle\sum_{j=1}^{n}\frac{B_1-B_j}{A_1A_j}\right)}{2(-2a)^{n-1}P_n(S_n-a)}$$

$$= \frac{\dfrac{T_1}{A_1} - \displaystyle\sum_{j=1}^{n}\frac{B_1-B_j}{A_1A_j}}{2(S_n-a)}$$

By a cyclic interchange of subscripts we have

$$u_i = \frac{\dfrac{T_i}{A_i} - \displaystyle\sum_{j=1}^{n}\frac{B_i-B_j}{A_iA_j}}{2(S_n-a)}, \qquad (i=1,2,\cdots,n).$$

Since the sums

$$\sum_{i,j=1}^{n}\frac{B_i-B_j}{A_iA_j}$$

are finite, we have by interchanging the order of summation

$$\sum_{i,j=1}^{n}\frac{B_i-B_j}{A_iA_j} = \sum_{i,j=1}^{n}\frac{B_i}{A_iA_j} - \sum_{i,j=1}^{n}\frac{B_i}{A_iA_j} = \sum_{i,j=1}^{n}\frac{B_i}{A_iA_j} - \sum_{i,j=1}^{n}\frac{B_i}{A_iA_j} = 0.$$

We have then

$$\sum_{i=1}^{n}u_i = \frac{1}{2(S_n-a)}\left(\sum_{i=1}^{n}\frac{T_i}{A_i} - \sum_{i,j=1}^{n}\frac{B_i-B_j}{A_iA_j}\right)$$

$$= \frac{T}{2(S_n-a)}$$

where

$$T = \sum_{i=1}^{n}\frac{T_i}{A_i}$$

Hence

$$p = \frac{T}{2a(S_n-a)} - \frac{b}{a} = \frac{T-2b(S_n-a)}{2a(S_n-a)}.$$

3. *Cournot Competition.* We consider now the problem of determining the number of units to be produced and the price per unit in order that each producer may make a maximum profit, assuming that his production is independent of the others. We have

$$\pi_i = u_i p - q_i(u_i), \qquad (i=1,2,\cdots,n).$$

Since

$$\frac{\partial \pi_i}{\partial u_i} = p + u_i \frac{\partial p}{\partial u_i} - 2A_i u_i - B_i, \qquad (i=1,2,\cdots,n),$$

it is necessary‡ that $u_1, u_2, \cdots, u_n$ satisfy the n equations

$$\begin{aligned} 2u_1(1-aA_1) + u_2 + \cdots\cdots + u_n &= b + aB_1, \\ u_1 + 2u_2(1-aA_2) + \cdots\cdots + u_n &= b + aB_2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ u_1 + u_2 + \cdots\cdots + 2u_n(1-aA_n) &= b + aB_n. \end{aligned}$$

Solving this system of equations we get

$$u_i = \frac{1}{1+R_n}\left(\frac{T_i}{M_i} + a\sum_{j=1}^{n}\frac{B_i - B_j}{M_i M_j}\right), \qquad (i=1,2,\cdots,n),$$

where

$$R_n = \sum_{i=1}^{n}\frac{1}{M_i} \text{ and } M_i = 1 - 2aA_i, \qquad (i=1,2,\cdots,n).$$

Since

$$\sum_{i,j=1}^{n}\frac{B_i - B_j}{M_i M_j} = 0,$$

$$\sum_{i=1}^{n} u_i = \frac{1}{1+R_n}\sum_{i=1}^{n}\frac{T_i}{M_i} = \frac{L}{1+R_n},$$

where

$$L = \sum_{i=1}^{n}\frac{T_i}{M_i}.$$

‡This is sufficient since $\dfrac{\partial^2 \pi_i}{\partial u_i^2} = 2(1/a - A_i) < 0.$

The price p, then, is

$$p = \frac{1}{a}\sum_{i=1}^{n} u_i - \frac{b}{a}$$

$$= \frac{L}{a(1+R_n)} - \frac{b}{a}$$

$$= \frac{L-b(1+R_n)}{a(1+R_n)}.$$

4. *Strict Competition.* We consider finally the problem when the n producers are operating under Strict Competition. In this case p is fixed. We have

$$\pi_i = u_i p - q_i(u_i), \qquad (i=1,2,\cdots,n),$$

and since

$$\frac{\partial \pi_i}{\partial u_i} = p - 2A_i u_i - B_i, \qquad (i=1,2,\cdots,n),$$

for a maximum π_i it is necessary* that $u_1, u_2, \cdots, u_n$ satisfy the n equations

$$u_1(1-2aA_1)+u_2+\cdots\cdots+u_n = b+aB_1,$$
$$u_1+u_2(1-2aA_2)+\cdots\cdots+u_n = b+aB_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_1+u_2+\cdots\cdots+u_n(1-2aA_n) = b+aB_n.$$

Solving this system of equations we get

$$u_i = \frac{T_i}{A_i(S_n-2a)} - \frac{1}{2(S_n-2a)}\sum_{j=1}^{n}\frac{B_i-B_j}{A_iA_j}, \qquad (i=1,2,\cdots,n).$$

Since

$$\sum_{i,j=1}^{n}\frac{B_i-B_j}{A_iA_j} = 0,$$

*This is likewise sufficient since $\dfrac{\partial^2 \pi_i}{\partial u_i^2} = -2A_i < 0$.

$$\sum_{i=1}^{n} u_i = \frac{T}{S_n - 2a}.$$

We have then

$$p = \frac{T}{a(S_n - 2a)} - \frac{b}{a}$$
$$= \frac{T - b(S_n - 2a)}{a(S_n - 2a)}.$$

THE Louisiana State University Press announces the forthcoming publication of *Carl Friedrich Gauss: Inaugural Lectures on Astronomy and Papers on the Foundations of Mathematics,* translated, composed and edited by G. Waldo Dunnington, of the University of Illinois. The appearance of this volume coincides with the celebration of the two-hundredth anniversary of the University of Göttingen, where a century ago the great physicist, mathematician, and astronomer was a member of the faculty. The book will be ready for distribution June 15; the price will be $1.00.

THOMAS FANTET DE LAGNY

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Notes on Thomas Fantet de Lagny

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I

When Thomas Fantet de Lagny lay dying and was no longer conscious of those about him, his friend Maupertius, to test if the spirit was still there, asked, "What is the square of 12?" "144", came the clear answer. They were his last words.

De Lagny was one of the lesser lights in a century great with mathematical luminaries of the first magnitude. But his work was by no means commonplace and his efforts took a direction away from the ordinary and beaten paths. He was the first to derive formulas for tan mx and sec mx directly from the right angle triangle, as he was the first to set forth in clear form* the periodicity of trigonometric functions; he was also the earliest one to make a success of using higher differences in solving higher degree numerical equations. He was a peculiar combination of the conservative and the iconoclast: in geometry, static, in algebra and trigonometry, adventurous, impatient. Withal he writes with a charm of style and with an orderliness of thinking not common among writers of scientific treatises in his period.

Now that Vieta, Descartes, Fermat, Pascal, Leibniz and other stars of the first magnitude in this century by their very stature have been acclaimed and studied, there is leisure to look into the less observed places for both men and methods. The biographer and the historian of mathematics may, if he will, find both interest and profit in getting acquainted with the lesser lights of this notable century,—Oughtred, La Hire, De Lagny, Ghetaldi, Dechales, etc. They lived interesting lives, did interesting things, suggested challenging problems, solved some of them.

*D. E. Smith, *History of Mathematics*, New York, 1925, Vol. II, p. 612.

Particularly interesting is T. F. De Lagny, as his productive period spanned half a century and his books and memoirs were published with uninterrupted regularity all that time. Clear, spontaneous, they enable the reader to imbibe the mathematical atmosphere of the period and to re-live, as it were, the mathematical growth of those decades.

II

He was born in Lyons, France, on November 7, 1660. His father, Pierre Fantet, was royal secretary of the chancellery at Grenoble; his mother, Jeanne d'Azy, was the daughter of a doctor of medicine in Montpellier† His early instruction was entrusted to his paternal uncle, the canon of Jouarre. Destined for the law he was sent to a Jesuit college in Lyons to study the classics. He was uncommonly alert and it is said that he wrote Greek verses while his comrades could as yet scarcely read Greek and that he wrote in Latin measures what was dictated to the class in French.

One day while yet in tender years he happened to see Pelletier's Algebra and Fournier's Euclid. These he bought and the reading of them awakened a joy in mathematics that became his for life.

In school work he always ranked first. His mind was so keen it seemed as if he was not learning at the lesson but was already divining what was in the next step. This gave him much spare time which he used for his own pleasure,—that is, to study mathematics; but in secret.

Deferring to his father's wishes he went to Toulouse to study law, and remained there for three years. If belles lettres was sacrificed for mathematics, he gave Philosophy even less consideration,—at least Philosophy as then taught. For while poetry and the classics were meant only to move the imagination, Philosophy pretended to reason. Equally he resisted the beckoning of the law, despite the rosy promises of M. de Fieubet, President of the Parliament. For he had found his field in mathematics.

After three years' study in Toulouse, when he was 18, he went to Paris, the mecca of mathematicians. Here was the Royal Academy

†Most of the biographical data are taken from Fontenelle's Eulogy in *Histoire et Mémoires de l'Académie de Sciences*, (after this labeled *Mémoires*), Paris, year 1735, pp. 107-114. Other sources are:

F. D. Frères, *Nouvelle Biographie Générale*, Paris, 1859, Vol. 28, p. 826;
Michaud, *Biographie Universelle*, Vol. 22, p. 519;
C.Hutton, *Philosophical and Mathematical Dictionary*, London, 1815, Vol. I p. 706.
Alex. Chalmers, *The General Biographical Dictionary*, London, 1815, Vol. 19, p. 455.

†He must have taken the name Lagny from the locality where he lived. For in presenting his little tract, *La Cubature de la Sphère*, in 1703 he signs himself simply "Thomas Fantet, *Lionnois*."

of Sciences, and he already had dreams of entering its portals. With only the equipment of the two books on mathematics which he had bought, with no teacher to guide him and no friends to discuss with, he yet had laid the foundation for some of the theories he was later to work out. Arriving in Paris he met the budding scientists's usual disappointment when in his reading he discovered that many of his theories had already been worked out by others, and yet, not all.

In 1682, working on the quadrature of the circle and searching for the "circumference-diameter ratio" he found the so-called Gregory series for π, being not then acquainted with the work done by Gregory and Leibniz on this same series. When he came to Paris he began to publish the results of his studies in the *Journal des Sçavans*. For a while, during his first stay in Paris, he was tutor to the Duke of Noailles.

In 1691 he published a work which created attention and which we shall discuss later, *Méthodes nouvelles et abrégées pour l'extraction et l'approximation des racines*. It dealt with a topic in which there was a lively interest among mathematicians and the treatise was discussed in various scientific journals of France and even in England. It secured for De Lagny his admission in 1695 into the Royal Academy of Sciences, the dream come true of the Toulouse youth who 18 years before had left for Paris. There have been few to whom Paris and its Academy of Sciences meant as much as to Thomas Fantet de Lagny. Throughout the grown-up years of his long life, from 18 to 74, it was the place where he felt it was good to be when there, the center to which his longing went when away.

In 1697 he published a text book in algebra, *Nouveaux Elémens d'Arithmétique et d'Algèbre*, one of the good text books of that period. The same year the Abbé Bignon, protector-general of letters, secured for him the post of professor-royal of hydrography at Rochefort. He was hesitant about accepting this position as he had no training in marine science. To allay his scruples he was permitted to make a long ocean voyage with a tutor so as to learn first-hand pilotage and marine science.

His duties were not onerous and he had leisure for study and writing. It was here that he wrote and published his two treatises on the cubature of the sphere (1702) and "binary arithmetic," (1703). It was also in these quiet years at Rochefort that he sent to the publications of the Academy three of his best researches, namely the "*Supplement to Trigonometry*" (1705), and his two notable memoirs on the solution of equations by aid of higher differences, in 1705 and 1706. He kept up his connections with the Academy by correspondence and sent his memoirs to be read to the assembly. He felt, however,

that his researches were not accorded the same attention they would receive if he reported them in person. And there may have been a reason for that. For, says Fontenelle, Perpetual-Secretary of the Academy, his ideas were so many and so new, so "impatient and quick," that they were not always arranged in "regular and tranquil order." De Lagny noticed, too, that he was losing interest contact with his fellow academicians. They were becoming much interested in the new geometry, analytical and projective, while De Lagny's studies dealt with the old classical geometry.

So he was anxious to get back to Paris. Just now France's desperate finances needed a mathematician. When the Duke of Orleans, Regent of France after the death of Louis XIV, in 1716 offered him the sub-directorship of the General Bank—much for the same reason that England appointed her great mathematician Newton master of the mint—he accepted. And so, says the witty Fontenelle, these two great geometers, contemporaries, who had been surrounded by the riches of ideas on every hand were now surrounded by the riches of silver on every hand. But England and Newton had the better luck. French money again- -! And the fruits of the seventy year reign of the Grand Monarch. (But Fontenelle does not put it that way.) In the crash of the Bank, De Lagny lost nearly all his private fortune.

But still he had his mathematics and his intellectual companionship in the Academy. On his return to Paris he had resumed his activities there and produced a memoir every year or two. The following thirteen years nine memoirs came from his pen, four on algebra, one on geometry, four on goniometry. The society rendered him a signal honor by devoting the entire publication for the year 1733 to a compilation and revision of his work in algebra, under the title of *Analyse Géneral.* It was to have been done by De Lagny himself but events made it necessary to give the work to his intimate friend, M. Richer. Unfortunately he did not do a good piece of editing. The clearness of style and orderliness of development in De Lagny's own presentation is lacking; and Richer committed the editorial sin of inserting many of his own ideas and hobbies.

In 1723 he was getting feeble and asked for and received a pension. He died in Paris, April 12, 1734. In 1714 he had given a paper on the cubature of portions of a sphere equivalent to rectilinear pyramids. In imitation of Archimedes he asked that these figures be the memorial on his grave.

De Lagny was said to be a cheerful and happy person. Also given to philanthropy; for though he never had a large fortune he

supported several poor families. In spite of his extravagant claims for his new methods he is described as a man of simplicity and modesty. He was married twice and had one child, a daughter from the first marriage.

III

De Lagny was a non-conformist. For mathematics has had its non-conformists also. Hardly in the logic of it: the nature of mathematics does not give much leeway there. But in processes, methods, purposes, emphasis.

The algebra inherited by modern Europe from Greek, Hindu, and Arab sources was largely a science of problem solving by means of the equation. The development of the theory of these equations and the perfecting of symbols was peculiarly the gift of the Renaissance and post-Renaissance centuries. Towards the end of the 17th century the delimiting of subject matter was practically accomplished and the new techniques were tending to stabilize. Methods, however, were laborious, form and exposition left much to be desired. For over thirty years, from the announcement of his new approximation methods in 1691 to the publishing in 1723 of his memoir on irrational quantities, he stressed the adoption of shorter and clearer procedures in algebra and trigonometry—sometimes on debatable ground—and continually threatened the *status quo* of methods. A good thing while these studies were in their formative stages.

De Lagny's first notable work to attract attention and the one that gained for him the admittance to membership in the Royal Academy of Sciences was *Méthodes nouvelles et abrégées pour l'extraction et l'approximation des racines.* It presented a general method of extracting the irrational roots of arithmetic numbers up to the fifth by approximation. He published a digest of his results in the *Journal des Sçavans* for May 14, 1691. The treatise itself was printed the following December and copies distributed to the members of the Academy of Sciences at their session on December 12, 1691. Copies were likewise sent to the members of the Academy living in England, Holland, Germany, and the provinces. This was done to find out if others were doing similar work in this field and to get their reactions.* It must have been from one of these copies that Edmund Halley got the garbled results that inspired him to work out his elegant approximations.†

**Journal des Sçavans* for 1691, p. 277; 1692, p. 181; *Mémoires*, year 1733, p. 336, *Méthodes Nouvelles*, preface.

†For Halley's work in approximating roots see M. A. Nordgaard, *A Historical Survey of Algebraic Methods of Aproximating the Roots of Numerical Higher Equations up to the Year 1819*, New York, 1922. pp. 41-44.

There were some errors of printing in the first edition, so a second press was made in June, 1692. It is a quarto of 62 pages, the first 43 pages explaining his abridged methods, the remaining pages applying them to the geometric problem of constructing solids and sursolids by means of ruler and compass, including the duplication of the cube.

The key to De Lagny's attitude in this and all his later works can be found in the opening sentences of his first book:

> "It is not enough for giving it to the public that a new method be sure and infallible, if it is not also shorter and simpler than the older methods. You would laugh at a guide who would offer to conduct you by a new road, but one which is much longer and more difficult than the ordinary road and where one runs a greater risk of getting lost. ..... Geometry and Algebra have their prejudices just like Physics and Medicine. And the one which I am going to discuss, [viz., the prejudice of Vieta's method] is one of the most firmly established."

After discussing current methods he proceeds with his own. Observing that the expression $z=\sqrt{a^2+b}$ has a rational form $z=a+\frac{b}{2a}$ and an irrational form

$$z=\frac{a}{2}\pm\sqrt{\frac{a^2}{4}\pm b}$$

as approximations, he obtains similar forms for $z=\sqrt[3]{a^3+b}$. These he found to be

$$\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b}{3a}}>z>\frac{a^2}{2}+\sqrt{\frac{a^2}{4}+\frac{b-1}{3a}}$$

and

$$a+\frac{ab+a}{3a^3+b+1}>z>a+\frac{ab}{3a^3+b}\ \ddagger$$

Thus:

$$z^3=a^3+b$$

$$z=\sqrt[3]{a^3+b}=a+x, \text{ where } x<1,$$

$$a^3+b=a^3+3a^2x+3a^3x+x,$$

$$\therefore\ b=3a^2x+3ax^2+x^3.$$

‡*Méthodes Nouvelles*, pp. 24-30; *Memoirs*, pp. 344-5

$x<1$, therefore x^3 is small. But suppose $x^3=1$, app.

Then $3a^2x+3ax^2+x^3=3a^2+3a+1$, app.

Now $3a^2x+3ax^2+x^3=b$.

Then $3a^2+3a+1=b$, app.

Disregard x^3, then $3a^2x+3ax^2=b$ app.

$$x^2+ax=\frac{b}{3a},$$

$$\therefore\ x=\sqrt{\frac{a^2}{4}+\frac{b}{3a}}-\frac{a}{2},$$

a little large, since for b we should have $b-x^3$ to be exact. But x is unknown and undetermined. However $x<1$, therefore $b-1$ makes the result a little too small.

$$\therefore\ x=\sqrt{\frac{a^2}{4}+\frac{b-1}{3a}}-\frac{a}{2}, \text{ a little too small.}$$

Therefore, since $z=a+x$,

$$\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b}{3a}}>z>\frac{a}{2}+\sqrt{\frac{a^2}{4}+\frac{b-1}{3a}}.$$

By similar processes he derived the rational forms. He also derived formulas for second approximations. As they are rather complex we shall only show how he derived a second approximation for square root.* Using Theon's formula,

$$\sqrt{a^2+b}=a+\frac{b}{2a}=c;\ (a^2+b)-c^2=d;$$

$$\sqrt{a^2+b}=\sqrt{c^2+d}=c+\frac{d}{2c}=a+\frac{4a^2b+b^2}{8a^3+4ab}.$$

He presses a point by showing how this method is expeditious also when the root is exact; in fact, is preferable to the Hindu digit-by-

**Nouveaux Elémens*, p. 304; *Méthodes Nouvelles*, pp. 20-21.

digit method. In the Algebra text that he published in 1697 he illustrates* this by extracting the cube root of 817400375, using the rational form

$$\sqrt[3]{a^3+b}=a+\frac{ab}{3a^3+b}$$

Thus: $$\sqrt[3]{817}=\sqrt[3]{729+88}=9\ \frac{792}{2275}.$$

As the original number needs three digits in the root, he multiplies

$$9\ \frac{792}{2275}\ \text{by 100 and gets 935,}$$

which, of course, is the exact root. Just so!

An enthusiastic review in *Journal des Sçavans*† says that by this method one can find the cube root of a number "in 2 or 3 minutes which by the (then) known methods would take several hours and even several days."(!)

As an illustration he finds the cube root of

959,985,256,734,003,407 to be 986479.

As follows:

$$959{,}985=98^3+b;\ b=18{,}793;\ \frac{98\times 18{,}793}{3\times 98^3+18{,}793}\times 10{,}000=6479$$

Therefore 980,000+6479=986,479 is the root.

Now the reader may time himself, using the Hindu method,—if one must have the cube root of such numbers.

Among the sporadic revivals of cube root approximations that have appeared in our school texts since 1900, the writer has seen some more clumsy formulas than his irrational formula used in this illustration from De Lagny:‡

$$\sqrt[3]{100}=\sqrt[3]{64+36}=2+\sqrt{7},\ \text{too large,}$$

$$=2+\sqrt{6^{11}/_{12}},\ \text{too small.}$$

**Nouveaux Elémens*, p. 473.
†May 10, 1692, p. 182.
‡*Méthodes Nouvelles*, pp. 42-43.

For the fourth root, if $z^4 = a^4 + b$

$$z = a + \frac{abc}{4a^4d + 2b^2}, \text{ where } c = 8a^4 + 3b \text{ and } d = c + 3b.$$

For the fifth root, if $z^5 = a^5 + b$, he derives the rational formula

$$z = a + \frac{abc^2 + 4ab^2c + 2ab^3}{c^3 + 6bc^2 + 8b^2c + b^3}.$$

and the irrational formula

$$z = \tfrac{1}{2}a + \sqrt{\sqrt{\tfrac{1}{4}a^4 + b/3a} - \tfrac{1}{2}a^2}.$$

De Lagny's formulas received much publicity and admiration. De Lagny himself thought highly of his invention. "I have no fear," he says, "in stating that never in the science of numbers has there been a discovery so beautiful in theory or anywheres near so useful in practice."

Approximation methods for obtaining roots of higher degree was a very live topic at this time. With the invention of analytic geometry and the calculus, the finding of roots became more than ever necessary,—not as a theoretic thing in the study of equations as formerly but as for actual use. Tartaglia's method of solving a cubic, so correct in theory, had proved almost valueless in actual use. Many such approximation methods had therefore been presented by several algebraists: Stevin, Vieta, Bürgi, Newton, Rolle, Raphson, and others. De Lagny's method was one of the more promising. Edmund Halley took up his suggestion in England and we find this type of method used in some of the school texts.*

One wonders whether, if De Lagny had lived in England, where calculational mathematics always has attracted greater attention than it seems to have done on the Continent, his work in the roots of numbers and his later work in the solution of equations would have gained greater vogue. It is interesting to notice that the two most used approximation methods of today, Newton's and Horner's, found an audience and hence gained a leverage in England. We note, too, the present day work in statistics and the Calculus of Observations emanating from London and Edinburgh.

*See W. Jones' arithmetic, *Synopsis Palmariarun Matheseos*, London, 1706, pp. 191-192.

IV

His next major work was an algebra, published in Paris in 1697 under the title, *Nouveaux Elémens d'Arithmétique et d'Algèbre.* Its chief contribution from a creative point of view was a new method of approximating the roots of higher numerical equations somewhat similar to the method shown in his *Méthodes nouvelle et abrégées* for the roots of numbers. The algebra is one of the good text books of this period. For readableness, for good psychology used in the introduction of new matter, for the offices of illustration, analogy, and drill, it is much superior to most English, German, and French texts of this period. De Lagny, in his books and memoirs, was a natural teacher. He loved mathematics; loved it so much that he did not feel satisfied till others should love it also. His text, both by contents and as example, must have been an influence in the schools, as must also his persistent striving for new and better methods and demonstrations in the Academy and in journals, "teaching the teachers." *Nouveaux Elémens* is a general text, and De Lagny's new methods of 1692 and 1697 are incorporated in the text along with the methods of Vieta, Descartes, Rolle, and others.

As an introduction to his new method* he solves the equation $x^3=60x+400$ by Tartaglia's method (and, humanly enough, he does *not* minimize the mechanical difficulties) where

$$x=(5+\sqrt{5})+(5-\sqrt{5})=10,$$

a rational root. Then he argues: 10 may be the root of 99 different equations of the form $x^3=ax+b$, namely:

$$x^3=1x+990$$

$$x^3=2x+980$$

$$x^3=3x+970 \text{ etc.}$$

$$x^3=99x+10$$

Of these 99 equations only 5 reach us under rational forms, namely:

$$x^3=27x+730$$

$$x^3=48x+520$$

$$x^3=63x+370$$

$$x^3=72x+280$$

$$x^3=75x+250$$

**Nouveaux Elémens*, p. 464.

The others present mechanical difficulties. But more than that, wherever $(\frac{1}{3}a)^3 > (\frac{1}{2}b)^2$ we have the irreducible case. "Exactly a fourth of all the equations of this formula involve imaginaries." We need then a process whereby we can obtain the exact or approximate roots of $x^3 = ax + b$ and $x^3 = ax - b$.

"Voici ce que j'ai trouvé la dessu."*

When $x^3 = ax + b$,

$$x = \sqrt[3]{\tfrac{1}{2}b + \sqrt{(\tfrac{1}{2}b)^2 - (\tfrac{1}{3}a)^3}} + \sqrt[3]{\tfrac{1}{2}b - \sqrt{(\tfrac{1}{2}b)^2 - (\tfrac{1}{3}a)^3}}$$

In the irreducible case $(\frac{1}{3}a)^3 > (\frac{1}{2}b)^2$, x is dominant and we can neglect b. Development:

$$x^3 = ax + b;\ x^3 = ax, \text{ app.; therefore } x = \sqrt{a}.$$

But this is too small, we must allow for b. Suppose $x = \sqrt{a} + y$; or for ease of calculation, let $x^3 = ax^2 + b$, where $x = a + y$. Then $x^3 = (a+y)^3 = a^3 + 3a^2y + 3ay^2 + y^3$. Also, $x^3 = a^2x + b = a^2(a+y) + b = a^3 + a^2y + b$. Therefore $3ay^2 + 2a^2y + y^3 = b$. Neglecting b, as being small, we have $3ay^2 + 2a^2y = b$, a quadratic in y being small, we have $3ay^2 + 2a^2y = b$, a quadratic in y.

Here
$$y = -\tfrac{1}{3}a + \sqrt{{}^1/_9a^2 + b/3a};$$
$$x = a + y = 2a/3 + \sqrt{{}^1/_9a^2 + b/3a}.$$

Further work in solving higher degree numerical equations was done by De Lagny in 1705 and 1706 when he successfully used higher differences to obtain both exact and approximate roots. J. Collins had suggested this approach 30 years earlier, but he did not do much with it. In 1723 De Lagny formed tables as an aid in this method of solving.†

It was in geometry that he carried on his earliest researches; after giving most of his productive energy in the middle years to algebra, he returned in his later years to his first interest. Besides the works on the quadrature of the circle and the cubature of the sphere referred to previously, he wrote a memoir in 1723 on irrational numbers, in which he attempts to reconstruct Archimedes' approximation of the value of π.‡ On the whole his work on classical geometry seems barren and not up to his work in algebra. But in one case he stood on the threshhold of a great discovery. In his quadrature of the circle in

*_Nouveaux Elémens_, pp. 473-474.
†_Mémoires_, year 1723, pp. 55-69.
‡_Mémoires_, year 1705, pp. 277-300; year 1706, pp. 296-318; year 1723, pp. 264-319.

1719 he stated, but could not prove it, that if the tangent to an arc is a rational number the arc must be irrational and the circle cannot be rectified.* It was this statement that suggested to Lambert his proof of the irrationality of π in 1767.

V

But if De Lagny did not succeed in bringing glamor to the past or extending the borders of classical geometry by implementing them with his new methods of algebra, he did valuable work by *using* that geometry as well as algebra in new and untried or partially developed fields,—cartography, angle measure, and, decidedly, analytical trigonometry. Not that the additions were numerous, but they were clear cut.

He opened up new thoughts on map drawing to meet the situation when it was discovered that the earth is not a sphere.† Though it had been previously suggested that a unit radius be employed in trigonometry instead of the sexagesimal 60 used since the time of Ptolemy, it was De Lagny's discussion of it in his quadrature of the circle in 1719 and in his three memoirs on Goniometry that gave the idea recognition.‡ In his *Supplément de Trigonométrie*** he developed for the first time formulas for tan mx and sec mx directly from the right-angled triangle. In his last 10 years he occupied himself largely with developing a new field which he called Goniometry, in which is taken up the various ways of measuring plane and spherical angles with the use of only the compass and an unmarked semi-circle.

One almost feels there should be an epitaph for a man whose works were written so spontaneously and published so uninterruptedly for forty years; who meets us mathematicians at the age of 18 trudging to Paris to join the Pythagoreans and leaves us at the age of 74 with the square of 12 in his dying breath.

One feels more interested in him than if he had not lived so long; for we have had time to live with him while pursuing his thoughts through the musty tomes of the Académie des Sciences. From the vantage point of 200 years we watch him struggle, bewilderedly, with the nature of π,—we who know the answer derived decades later. We see him attack new problems in cartography as the increasing knowledge of the globe brings new difficulties,—not difficulties to us who have 200 years of experience to go on. Other problems and other

*_Mémoires_, year 1719, p. 141.
†_Mémoires_, year 1703, pp. 95-101.
‡_Mémoires_, year 1791, p. 144; year 1724, p. 292; year 1725, p. 284; year 1727, p. 121.
**_Mémoires_, year 1705, pp. 254-263.

efforts; all reflected in those pages. We look at his little *Méthodes nouvelles et abrégées* of 1692, his algebra of 1697, and then memoir upon memoir, lucidly written, moderately heavy in content, until 1729. Then 1730, none; 1731, none. One fears there will be no more, for 1734 is near. And there is no more. We open the 1734 volume,—there is Fontenelle's 8 page *Eloge*. We seem to have been watching the life history of an intellect.

The Sources of Euclid

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The title above is not an example of the sort of variation which English of the common kind sometimes exhibits when lifted from a place of obscurity to the head of an article. Euclid is not merely a man, but either a man or a book. The paragraphs below are rather about the subject matter of the book, than about the texts consulted by its author. Euclid's sources are those works upon geometry to which he had access; whereas the sources of Euclid are those divisions of knowledge from which geometry has chosen statements to include in its system of deduction.

Of course something can be said about Euclid's sources. Extant references to pre-Euclidean books of elements mention three. As all of these have long since disappeared their scope is a matter of guesswork, and information regarding their contents has to be derived from the remarks of other authors. Though few, such comments show that an arithmetic in Euclidean shape existed at least fifty years before the period of Euclid's activity, and that the proposition form used by Euclid was well established long before his time. In fact it is more than possible that long passages in the Elements repeat without substantial change some earlier work. For his references to geometry Aristotle probably made use of the last pre-Euclidean treatise on the elements. His quotations make it additionally clear that Euclid wrote in a manner already traditional, and made extensive use of earlier collections.

Other evidence that the Elements must be a compilation is presented by the structure of the book itself. For instance only a compiler can afford to spend his time defining notions which he never uses. If Euclid had been rewriting the elements of geometry, it seems unlikely that he should have assumed the task of presenting the subject of proportion twice, the more general treatment in Book V and subsequently another in Book VII. The second of these is probably due to the Pythagoreans both in substance and in form, while the first at least in substance is the work of Eudoxos. Indeed an author entirely rewriting the elements would have found it most saving of his efforts to place Book V prior to much of Books I and IV.

Moreover certain sections of the Elements display rhetorical peculiarities marked enough to have withstood the erosion of twenty centuries of redaction and translation. Phrases so common in some of the books as to be characteristic of their style are scarcely to be found at all in others. The processes of addition and subtraction furnish some striking contrasts of this kind. The expressions "be added" and "be subtracted" occur with much the same frequency in all the books. The term "remainder" is likewise found throughout, although in the first four books its place is taken more than half the time by some circumlocution ending with the words "which remains". Variety of the widest is to be seen in the use of the word "sum." It does not occur at all in the first four books, the notion there being represented by a curious hybrid—the nearest approach to a symbol to be found in the text of the elements—which in Book I, theorem 47, takes the form "the square on *BC* is equal to the squares on *BA,AC*." In the remaining books, Book X alone excepted, the word "sum" is rare, being used in general less and often much less than its earlier substitute. In Book X it is used constantly, and much exceeds in frequency all equivalent expressions. Some critics have regarded Book X as the most recent in form, and the only one of the thirteen which Euclid wrote word for word. Another peculiarity of its style is its substitution of the word "divide" for the word "cut" in describing the location of a point interior to a segment. A second book with marked individuality is Book V, but there the unique material treated is reason enough for its style.

The view that Greek geometry is an outgrowth of Egyptian land surveying is traditional. It is, in other words, the traditional view that Egyptian land surveying is the source of Euclid. It has even been proposed that geometry was an Egyptian invention, and some attempts, supported principally by speculation, have been made to distinguish the Hamitic core of Greek science. Of course much honest doubt obscures the extent to which the mature growth was rooted in the soil of Africa. The question is complicated by the fact that the only extensive early record of Egyptian mathematics antedates by one thousand years the emergence of anything like it upon the shores of the Aegean. Even those whose attitude toward classic civilization is idolatrous concede the land of the Nile something, although there are instances in which only a miracle could explain its pigmy size. The story goes that the Nile by its periodic overflows and occasional changes in course made a system of surveying essential to the stability of boundary lines. It may be doubted whether the obliteration of landmarks was due really to the Nile rather than to human nature.

However Egypt was an agricultural land thickly settled, and no doubt needed surveyors.

The view that Greek geometry developed naturally from Greek surveying has never been advanced. A notable reason for this is the fact that its earliest figure was a man who had travelled widely in Asia Minor and Egypt. His name was Thales. He was really an engineer and inventor, not a geometer, and his exploits are properly a part of the sources of Euclid. Even these would soon have been forgotten but for one thing. In the year 585 B. C. he signed his name upon the register of history by predicting an eclipse. A visit to Babylon and not a visitation of genius seems to account for the feat. Neither Thales nor the Babylonians had any idea of the true cause of eclipses. But the Babylonian astronomers, with seven centuries of recorded observations at their disposal, had noticed the tendency of such events to occur in a cycle, in accord with which they made predictions. Often of course an eclipse predicted by the cycle could not be seen in Mesopotamia. In accord with the cycle Thales predicted an eclipse. Luck was with Thales; the eclipse occurred. Thus by the connivance of heaven invested with significance, his activities became a matter of record, whose surviving fragments are of great interest. From them some estimate of the status of geometry during his day can be made. From them it is inferred that geometry was still a study without logical structure.

If tradition were silent the picturesque terminology of the Elements would itself dispose the reader to infer an early relation between geometry and surveying. Constructions frequently include the words "be set out", as if with a mallet and stakes, "be joined" as if with an actual tape line, and "be carried through" or "around" thickets and quagmires. And yet as these formations recur from page to page, alternating with others, such as to be "on the same base" or "under the same height", to "stand on" or "be set up at", and to "fall on" or "between" or "awry", the earlier impression fades. Such words suggest rather the mason handling level and plumb line, the architect laying out a building, or even in the case of "be set out" merely a draftsman with T-square and thumb tacks. Whether in Egypt or in Greece the contractor, architect, artist, and artisan have surely had as much to do with determining the content of geometry as the surveyor; and the history of art makes it easy to believe that in Greece at least they may have had much more.

The architect or artisan planning his work is confronted by problems geometrical in aspect, the vital one of which is that of filling a rectangle. Other shapes he meets, of course, but if he can't even

deal with a rectangle he may as well at once turn art critic. The facade of the building, the wall panel, the piece of canvas, the contour of the vase—these are his blank pages. What do to! Well, here goes—figures and frets, inscriptions, colors—striving all the while to make the constituents of the composition appear consistent—depending for all they are worth upon technical harmonizers such as uniform treatment, unifying colors, similar shapes. Yes, shapes again; rectangular shapes within rectangular shapes; rectangular shapes divided into other rectangular shapes. Now there are some rectangles which are more readily divisible than others into consistent shapes, into squares, for instance, and rectangles similar to themselves and in various positions. Artists who have noticed this make the achievement of unity a little easier for themselves by adopting such rectangles, and utilizing their natural subdivisions in locating salient features of design.

Of course any rectangle can be divided by a lattice of parallel lines into rectangles similar to itself. This type of division is too monotonous to be of much importance. Consider instead the following: Let A be a square whose base has length a, and upon the same base as the square let B and C be rectangles whose altitudes are respectively a times b and a divided by b. The area of the square is a mean proportional between the area of the rectangles, and each diagonal of the one rectangle is perpendicular to one of the diagonals of the other. It is clear that the ratio of the area of the two rectangles will be a natural number k, if b equals the square root of k. If k is 2, the perpendicular bisector of the long side of the rectangle B divides it into two rectangles similar to B and identical with C. If k is 2, the difference between the area of B and that of A is another square and a rectangle similar to B. These two subdivisions of the area of B may be used as a key for the resolution of B into shapes harmonious with B itself. When other values are assigned to k, or other conditions imposed upon b, other rectangles result whose areas admit similar graphing. A striking property of such rectangles is the fact that rarely are their length and breadth commensurable.

It may appear unlikely that many artists can have ventured far enough into geometry to find ready use for such diagrams as these. Appearances, however, controvert this view. The artists of Greece, following at least in point of time those of Egypt, made constant use of some of the special rectangles in question. A surprising body of measurements is evidence of this. Of special interest and completeness are those which have been made of the Greek vase. The potters of Greece were prolific craftsmen many of whose works survive today

in perfect condition undamaged and unrestored by that vehement admiration which has squeezed more ambitious examples of Greek art so badly out of shape. Vases being easy to handle have especially invited scrutiny of their dimensions, and what care has been expended in this pursuit reveals a widespread use of irrationalities in Greek vase form. The most accomplished of the potters worked between 600 B. C. and 450 B. C., after the later date a rapid decay in quality soon reducing their output to trash. The earliest of the great vase makers must therefore have died about the time Pythagoras was born, and the last about a century before the birth of Euclid. One hundred and fifty years before the first book of Elements, considerable information concerning the geometrical properties of the simpler irrational numbers had been recorded in the shapes of vases. Of course it was very special in character. There is very little reason to believe that the architects of the Greek vase ever allied it with anything more involved than the square root of five. That they did even that is astonishing enough. It is difficult indeed to believe that the traditions of the potters guild had nothing whatever to do with the content of Books II and X. By mathematicians who now look back upon the third century B. C. along the highway which their science has traveled since that time, these books are called a geometrical algebra. It is nevertheless entirely possible that their content was to the Greeks no more algebraic than their form is to us. The excesses and deficiencies, the binomials and apotomes of their elaborate logic, always so intimately associated with areas, are surely as close kin in spirit to the manipulation of forms which was Greek as to the manipulation of symbols which was not. Whereas upon the one hand to regard the geometrical algebra as a handbook of instruction upon the principles of Greek aesthetics would deserve no more than laughter, to regard it on the other as without sources in the fine and applied arts would be willful disregard of the signs.

Accumulations of information of the sorts noted above were available to scholars during the sixth century B. C. Among those scholars was one who first observed that between certain items of this store there were inferential relationships. To whom the notion first occurred can not be said, but it seems to have been fairly familar to Pythagoras and fairly unfamilar to his predecessors.

In spite of the alleged superiority of the Greek social order, its mathematicians were no more prominent during its heydey and in its records than those of later and less illustrious times. Concerning the lives of even such men as Eudoxos and Euclid the historians of today are forced to confine themselves to a page or so, much diluted by dis-

agreements with their authorities. It is thus a relief to find circumstances entirely different in the case of Pythagoras, where information is if anything too plentiful. He was, it seems, already a universally learned man when he left his early home in Samos to complete his education. This he did on the one hand by spending twenty-five years in Egypt, and on the other not more than two, finding out from the Egyptian priests all that they knew, although he could neither read nor understand Egyptian. Thus prepared for almost everything, he spent the remainder of his life as a simple man of science devoted to the discovery of truth, which he exemplified by conversing with a voice from heaven, making simultaneous appearances in two different towns, and killing a deadly snake by biting it. If it were not for the fact that among his followers and admirers are all of the scientists and many of the philosophers to be found in Greece during a century and more, circumstantial evidence accepted at its face would label him a thorough quack and humbug. At Kroton in southern Italy, where he settled sometime after 529 B. C., he passed the most significant part of his career. Here he became the principal figure in a religious brotherhood whose purpose was the moral reformation of society. It was not essentially either a philosophical school or a political league, and seems to have been democratic in its ideals. At first apparently it established a considerable degree of control over Magna Graecia, and this involved it in conflict. The destruction of Sybaris in 510 B. C. brought opposition to it into the open, and Pythagoras himself retired to Metapontion, where he died about the year 500 B. C.

The order remained powerful in southern Italy for half a century longer. After 450 B. C., however, violence was its portion, its lodges were burned, many of its members ambushed and assassinated, and those in whom judgment was a truer adviser than courage made the the best of opportunities to escape to Greece itself. Although a few of them later returned to Italy and became prominent, the society as a whole had lost its coherence, and as a philosophical school was extinct by the end of the century.

The intellectual pattern of the order is difficult to trace, as oral transmission was the custom of the clan and persecution interrupted its continuity more than once. It seems, however, to have included immortality and the kinship of all living creatures. Cosmological suppositions were of course an early part of it, of which the sphericity of the earth is credited to the founder. Originally the Pythagorean rule included a number of picturesque taboos designed to gratify in their observance the spiritual cravings of the less intelligent. At a

later date the kernel of its philosophy as reported by Plato is this:—the body is the tomb of the soul—from this tomb release is essential—knowledge provides the surest release.

The deification of wisdom inherent in this creed began with Pythagoras himself, and determined the principal achievements of its exponents. At no time were they content with a mere storing of information. It was their aim to add to knowledge and organize it. Observation and possibly even experiment, with vibrating chords for instance, had convinced Pythagoras of the fundamental significance of numbers. These he thought of as patterns of points, and in such diagrammatic forms regarded them as determining the properties of those elements which compose the universe. Resemblances between this attitude and that of modern chemistry are close enough to be surprising, but it is not clear how near the Pythagoreans ever came to an atomic hypothesis. Together with regular patterns in the plane, the corresponding regular patterns in space thus became objects of critical importance. It is no wonder that, related as they thus are to its religion, the properties of the regular polygons and the regular solids were matters of such absorbing interest to the Pythagoreans.

Can the content of geometrical theorems determine the form of religious beliefs? Can the content of religious beliefs determine the form of geometrical theorems. Does it mean anything to ask such questions in connection with the doctrines of a long vanished sect of tedious mystics, even if their cosmology and philosophy has been preserved without due credit in the writings of Plato? And yet if not, the impression, once entertained, that geometry in its early stages was fashioned by religious fervor is difficult to discard. That fervor—taking form haltingly first in Books I and II and then in Book IV (the one about regular polygons), inspiring Theaitetos (who with Plato studied under the Pythagorean Theodoros) to write the first treatise upon the regular solids revised by Euclid as Book XIII. Taking form again in several guises, as a theory of similar figures in Book VI and a theory of proportion in Book VII, transmitted by Archytas the Pythagorean and Theodoros to Theaitetos who wrote the first work upon irrationals (those devils of the Pythagorean theology) and to Eudoxos (whose teacher was Archytas), and completed at last as Books V and X. That fervor—has it not itself been another source of Euclid, supplying power if not substance as well?

So from the ingredients engineering, art, and religion the geometry of Euclid is compounded. The men who shaped it were a strange group of mystics. The Elements are their book. Of course they did

not write it themselves. Pythagoreas wrote nothing, his students wrote nothing, the society left no first hand account of its affairs. Those, not themselves Pythagoreans, who had been students of members of the group wrote the works which Euclid rearranged. But then the disciples of Christ did not write the New Testament, not even the Gospels—unless of course you prefer to feel romantic about the the Book of John—so it is no new thing that the holy book of a faith should be recorded by the hands of sympathetic bystanders. Like other volumes of sacred writings it has been the cause of varied and unintended consequences. It is hard to imagine what the science of today might be like if no one had ever taken the trouble to write down the bible of the Pythagoreans.

NOTE: The author of these pages sympathizes thoroughly with those writers of non-technical articles upon mathematical subjects, who courageously avoid a specious air of scholarship by abstaining from the use of footnotes. Having at times, however, wished in vain for further support of their more striking statements, he here lists the principal sources of his own:

A History of Greek Mathematics, by T. Heath.
Early Greek Philosophy by J. Burnet.
The Thirteen Books of Euclid's Elements, translated by T. Heath.
Dynamic Symmetry—the Greek Vase, by J. Hambidge.

The Teachers' Department

Edited by
JOSEPH SEIDLIN

Concerning the Teaching of the Linear Equation

By D. H. RICHERT
Bethel College, Kansas

The linear equation in two variables, x and y, may be written in the form

(1) $$y=px+q.$$

What rôle do p and q play in this equation? To show what effect they have upon the behavior of the graph of 1), let the student construct the graph when q has been given a fixed value and p is allowed to run through any set of values; next, assign to p a fixed value and let q vary. This exercise will show clearly what effect each of the parameters has upon the position of the line in the plane.

A surprising result is obtained when one considers an equation of the type

(2) $$a^2+ax+y=0,$$

here only one parameter is involved, but it occurs in the second degree. What is the meaning of such an equation?

If to a are assigned the values

$$\tfrac{1}{2},\ 1,\ 2,\ 3,\cdots,\ -3,\ -2,\ -\tfrac{1}{2},\cdots,$$

the graph of 2) yields the system of lines as shown in the figure.

Let a and $a+\Delta a$ be two neighboring values of a, then the equations

$$2)\quad a^2+ax+y=0$$

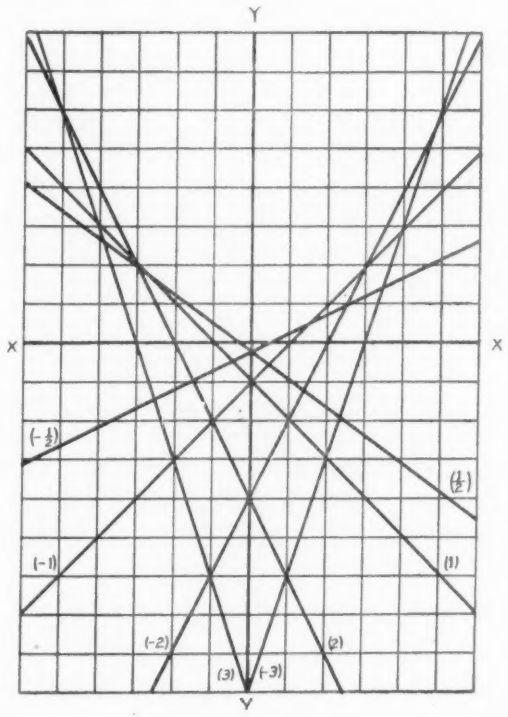
$$3)\quad (a+\Delta a)^2+(a+\Delta a)x+y=0$$

represent two neighboring lines. Equation (3) reduces to

$$y+a^2+ax+2a\cdot\Delta a+\Delta a^2+\Delta a\cdot x=0.$$

Since by (2) the sum of the first three terms is zero, the sum of the remaining three terms is also zero, hence we may write

(4) $$2a+\Delta a+x=0.$$



Equations (2) and (4) determine the point of intersection of their graphs. Letting $\Delta a \to 0$, the ultimate point of intersection of these graphs is determined by the equations

(2) $$a^2+ax+y=0$$

(5) $$2a+x=0.$$

We may look upon (2) as representing a family of an infinite number of tangents to a *curve* which is evidently the locus of the points of intersection of these lines. The equation of this curve may be obtained by eliminating the parameter a between equations (2) and (5), the result is the conic

(6) $$y=x^2/4.$$

From the above considerations it appears that every line whose equation involves one parameter in the second degree is a tangent to a conic.

We have said that (2) represents an infinite number of lines, but it should be noted that they do not fill the whole plane since none of the lines extent into the region within the conic (6).

On the other hand, equation (1) represents a doubly infinite set of lines, each set filling the whole plane. Construction of graphs as suggested above will help to enliven the whole subject.

The next thing a student might do in connection with this subject is to ask what phenomena follow straight line trends. If he should attempt to make a list of them from his study of the sciences, he would be surprised at the endless number of instances in which the relation between variable quantities can be expressed by means of a linear equation.

A Report of the Mathematics Committee of the California Junior College Association

Review by JOSEPH SEIDLIN

A committee of junior college mathematics instructors appointed by the California State Department of Education made and published a survey of the mathematics courses taught in the public junior colleges of California. The report is an exhaustive document for which the committee deserves commendations as well as honorable mention. The personnel of the committee is as follows:

J. Calvin Funk, Santa Maria Union Junior College (Chairman).
Laurence E. Anderson, Los Angeles Junior College.
Hope A. Jordan, formerly of California Polytechnic School.
C. Leslie Nichols, Glendale Junior College.
Henry W. Stager, formerly of Salinas Junior College.
Theron S. Taber, Jr., Bakersfield Junior College.

The report is of highly practical nature in that it deals with such prosaic matters as:

(1) *The "quantities" of various courses.* (See Table 1)
(2) *Variety and number of sections of 2-unit courses.* (See Table 2).
(3) *Student enrollment and mortality in all courses.* (See Table 3).
(4) *Simple, direct, and immediate objectives "ranked".* (See Table 3A).
(5) *Extent and variety of "terminal courses."*
(6) *Provision made for students who enter junior college without any secondary school mathematics.* (See Table 7).
(7) *Student enrollment and mortality in terminal courses.* (See Table 8).
(8) *Choice of textbooks in university transfer courses.*

In the analysis and conclusions the committee makes some terse statements of observation and asks some questions. Presumably all teachers of junior college mathematics are urged to supply adequate answers to these questions. It may be that the fate of the teaching of elementary college mathematics is bound up with the answers to

these questions. A note of disappointment, if not actual pessimism is clearly in evidence in the committee's analysis of the present status of the teaching of junior college mathematics in California. And certainly in the matter of junior colleges California is not among the "backward" states. The report, then, is of more than local significance.

One of the disturbing highlights is that the number of junior college students studying high school mathematics is alarmingly great. In this connection the committee makes the following comment and queries: "... high school mathematics courses, it seems, should be studied in high school and not be postponed until junior college, where they are a too frequent source of disturbance. If our high school teaching is faulty and if that accounts for so many of our high school students sidestepping algebra, geometry, and trigonometry, what can be done to improve such teaching? If the implication and supposition are debatable, where would the various high school courses in mathematics be taught to best advantage?"

But do the junior colleges show the high schools that "it can be done"? Well, hardly. For, continues the report, "Incidentally (?) the various algebra courses furnish the field in which a considerable number of students is eliminated from university transfer mathematics."

Of legitimate concern to the committee, and well may it be to all of us, is the subservience of junior colleges to universities. When did the junior colleges forfeit their best selling point,—the curriculum of terminal courses? How shall we account for the following state of affairs? "Comparing the number of students in university transfer courses with the number of students in terminal courses, the latter number is unfortunately small in comparison... The results indicate that outside of several large institutions little or no terminal work was offered...". According to the committee the evidence is clear-cut that there is "... little concern given to one of the important, if not the most important, raison d'etre of the junior college." Too timidly, it seems, the committee suggests that "With the additional schooling which the new economic situation has undoubtedly made necessary, the junior college must prepare to meet the needs of many other than university preparatory students."

In analysing the nature of the problems suggested as "applications of mathematics to other fields" the committee finds that "The problems to be considered seem to be limited almost entirely to the physical sciences and the tools to the well worn ones of many generations. Problems of a social nature and the modern tools with which to solve them seem to be placed out of reach of the youth who

are living in this modern society, groping in the dark for a way to preserve its very existence. Would it not be well for our instructors to get in touch with these applications to new fields... We believe it would give new life to our courses." And, one is tempted to add, save the old courses from senility dementia.

Let us get what mental balm and satisfaction we can out of "It is perhaps a good sign to observe that the junior colleges are independent in the choice of their textbooks in the university transfer courses. There seem to be no indications that they are influenced materially by the choice of textbooks by the universities."

In conclusion, as throughout the body of the report, there is the repeated claim that the junior colleges, even more brazenly than political parties, ignore the "planks" on which they were built. Once more, then: "The number of terminal courses is surprisingly small as compared with the university transfer courses... Since the junior college is supposed to be serving principally the community, it seems that it should offer more terminal mathematics courses for the youth not planning to go to college... Hence it is up to us mathematics teachers not only to find time during these crowded days to develop terminal mathematics courses, but, what is harder, to sell such courses to the youth of the community. Could we junior college mathematics teachers band together to promote this part of our work?"

I am reasonably convinced that the committee is sincere in propounding the above question and hopeful of a favorable reply.

In what quantities do the various courses occur? Table No. 1 gives this information.

TABLE No. 1

Number of Sections in the Various Quantity Courses
(University Transfer Courses)

3-unit courses comprise	300 sections
2-unit courses comprise	35 sections
5-unit courses comprise	11 sections
1-unit courses comprise	6 sections
4-unit courses comprise	3 sections
6-unit courses comprise	2 sections

TABLE No. 2

Number of Sections of Various 2-unit Courses
(University Transfer Courses)

Trigonometry (plane) as a 2-unit course	—12 sections
Differential Equations as a 2-unit course	— 6 sections
Solid Geometry (H. S.) as a 2-unit course	— 6 sections
Intermediate Algebra as a 2-unit course	— 3 sections
Supplementary Mathematics as a 2-unit course	— 2 sections
Monography, Victors, etc., as a 2-unit course	— 2 sections

College Algebra as a 2-unit course..... — 2 sections
Hyperbolic Functions as a 2-unit course..... — 1 section
Analytic Geometry as a 2-unit course..... — 1 section

How do the various courses rank with respect to the number of students enrolled in them? Table No. 3 contains the data.

TABLE No. 3

Number of Students Enrolled in and Number of Students Completing the Various Courses; e. g., in Analytic Geometry 1044 Students were Enrolled, but 876 Completed the Course.

(*University Transfer Courses*)

Name of Course	*Number of Students Enrolled*	*Number of Students Finishing Course*
Analytic Geometry	1044	876
Plane Trigonometry	803	655
College Algebra	779	674
Intermediate Algebra	745	615
Differential Calculus	681	578
Integral Calculus	639	558
Mathematical Theory of Investment	440	430
Elementary Algebra Theory	432	356
Mathematical Analysis 3A	241	214
Mathematical Analysis 3B	166	150
Solid Geometry (H. S.)	144	126
Spherical Trigonometry	119	95
Slide Rule	108	97
Surveying	107	105
Mathematics 14A	102	96
Mathematics 14B	102	93
Mathematics 10A	47	43
Mathematical Analysis 4B	34	33
Mathematics 10B	29	28
Mathematical Analysis 4A	25	22

TABLE 3A

a. Ability to solve any triangle in plane trigonometry.
b. Appreciation of significant figures and accuracy in computations.
c. Ability to use the Mannheim (simpler) type of slide rule.
d. Ability to use common logarithms.
e. Proficiency in algebraic fundamentals.
f. Ability to express physical situation in algebraic form.
g. Appreciation of the idea of fundamentality.
h. Ability to construct and interpret graphs.
i. Knowledge of general methods of attack for the solution of problems.
j. Appreciation of reasonable results.
k. Ability to apply the compound interest law to problems in finance in the broader sense of the term.
l. Ability to compute and interpret the principal terms and measures in connection with the curve of distribution as applied to statistics.
m. Ability in the determination and use of various forms of averages.
n. Ability in the construction, interpretation, and use of index numbers.

The fourth question considers the provisions made to take care of students who enter junior college without any algebra or plane geometry, but who wish to undertake further non-academic mathematics. Three methods were presented as follows.

instructors being asked to rank them in order and also to indicate the method or methods used in their own institutions:

a. Standard course in these subjects for those who lack them.
b. Special pick-up course to make up deficiency.
c. Permit the student to enter mathematics listed above directly.

The results from this question are presented in Table No. 7 in essentially the same manner as the results from the second question are given in Table No. 6.

TABLE No. 7

Method	Methods Preferred By Instructors		Methods Used			
			By Instructors		By Schools	
	Average Rank	Ranking	Number	Ranking	Number	Ranking
a (1)*	1.71	1.5	11	1.0	6	1.0
(2)	1.53	1.0	41	1.0	7	1.0
(3)	1.57	1.0	52	1.0	13	1.0
b (1)	1.71	1.5	0	3.0	0	3.0
(2)	1.77	2.0	12	2.0	0	3.0
(3)	1.76	2.0	12	2.0	0	3.0
c (1)	2.57	3.0	3	2.0	2	2.0
(2)	2.71	3.0	7	3.0	1	2.0
(3)	2.68	3.0	10	3.0	3	2.0

*(1) Departmental Junior Colleges.
(2) District Junior Colleges.
(3) All Junior Colleges.

TABLE No. 8

Number of Students Enrolled and Number of Students Completing Various Terminal Courses; e. g., in Engineering Mathematics 859 Students are Enrolled, While Only 687 Complete the Course.

Name of Course	Number of Students Enrolled	Number of Students Completing Course
Engineering Mathematics	859	687
Intermediate Algebra	597	457
Business Mathematics	572	448
Elementary Algebra	492	365
Mathematical Analysis	294	258
Advanced Algebra	226	196
Plane Trigonometry	214	163
Technical Mathematics	174	147

Table No. 8 shows that by far the most students are enrolled in engineering mathematics. Then follow intermediate algebra and business mathematics. The engineering mathematics outdistances the other courses easily. Should that be so? Are there not enough occasions for applications of mathematics in the field of business, business administration, investment theory, elementary statistics, and accounting, among others, to warrant more students in these lines as compared with engineering?

Mathematical World News

Edited by
L. J. ADAMS

Dr. E. J. Townsend has presented his extensive mathematical library to Albion College (Michigan) from which he received the Ph. B. in 1890 and LL.D. in 1915. He received the Ph.D. at Göttingen in 1900. Professor Townsend was appointed to the faculty at the University of Illinois in 1893; he served for many years as head of the department of mathematics there, retiring in 1929, and from 1905 to 1913 was dean of the College of Sciences.

Professor G. A. Miller was made an honorary life member of the Mathematical Association of America at the recent Durham meeting, having served as its president in 1921.

Professor E. R. Hedrick will not teach a summer course at Northwestern University, Evanston, Illinois, as previously announced. Professor N. J. Lennes will take Professor Hedrick's place on the summer school program. Professor Hedrick asked to be relieved of his summer school class because of his increased duties in his new position, provost of the University of California at Los Angeles.

Professor W. M. Whyburn is acting head of the department of mathematics, University of California at Los Angeles. He is filling the vacancy created when Dr. E. R. Hedrick was appointed provost at the university.

Twelve mathematicians held a two day symposium, April 7 and 8, at the University of Notre Dame, on *Calculus of Variations* and and its application to economics and physics. Among many practical applications, the streamlining of automobiles and airplanes and railroad equipment is the most important because this branch of mathematics computes that form of surface which will provide minimum air resistance. In economics, for the plotting of production curves over long periods of industry, calculus of variations supplies the key for planning.

Participating in the discussion were Marston Morse of Princeton; Gilbert A. Bliss, Lawrence A. Graves, and William T. Reid of the University of Chicago; Tibor Rado, Ohio State University; Walter Mayer, Princeton; Summer B. Myers, University of Michigan; Edward

J. McShane, University of Virginia; Karl Menger and Arthur Haas of Notre Dame; Lothar W. Nordheim, Purdue, and Charles F. Roos of New York City, former mathematical economist adviser to the NRA.

Professor H. H. Downing, University of Kentucky, reports the following news item from that institution:

1. Professor E. L. Rees, on leave for 1936-1937, has spent most of the year traveling in Australia and South Africa, and cruising about the southern Pacific.

2. Professor D. E. South, on leave for 1936-1937, is working on his doctorate at the University of Michigan.

3. Miss Sarah Pence, on leave second semester 1936-1937, is working on her doctorate at the University of Illinois.

4. Mr. Charles W. Williams, who received the master's degree at Harvard, June, 1936, has a graduate assistantship for 1936-1937.

5. Mr. W. H. Spragens, A. B. (Ky.), 1935, has a graduate assistantship for 1936-1937.

6. W. H. Pell, A. B. (Ky.), 1936, has a graduate assistantship for 1936-1937.

7. Mrs. Aughtum Howard, A. B. (Georgetown, Ky.), 1926, has a graduate assistantship for the second semester, 1936-1937.

8. Misses Pauline Thompson, Ruth Weatherford, and Elizabeth Johnstone are due to receive their bachelor degrees in mathematics, June, 1937.

9. During the summer, 1937, courses in advanced mathematics will be offered as follows:

First Term: June 14 to July 17
Vector Analysis, Dr. Le Stourgeon.
Advanced Analytics, Dr. Latimer.
Curve Tracing, Dr. Boyd.

Second Term: July 19 to August 21.
Solid Analytic Geometry, Dr. Downing.
Theory of Functions of a Complex Variable, Dr. Downing.

The forty-third Summer Meeting and twentieth Colloquim of the American Mathematical Society will be held at the Pennsylvania State College, September 7-10, 1937, in conjunction with the Summer Meeting of the Mathematical Asssociation of America. The Colloquium Lectures on *Continuous geometry*, to be given by Professor John von Neumann of The Institutute for Advanced Study, are tentatively

scheduled for Tuesday afternoon, Wednesday, Thursday, and Friday mornings. By invitation of the Program Committee, Professor Hassler Whitney of Harvard University will give an address entitled *Topological properties of differentiable manifolds.*

Dr. Fred W. Sparks and Dr. R. S. Underwood, professors of mathematics at Texas Technological College, attended the meeting of the southwest section of the Mathematical Association of America on April 2 and 3. Dr. Sparks presented a paper on *College Mathematics and the New Curriculum.* This meeting was held at the New Mexico State College of Agriculture and Mechanic Arts, State College, New Mexico.

On April 17 the regional meeting of the Texas Academy of Science was held in Lubbock, at the Texas Techonological College. Dr. R. S. Underwood, professor of mathematics presented a paper on *The Relative Delicacy of Certain Convergent Tests.*

Dean R. D. Carmichael of the University of Illinois was guest speaker at the 15th annual meeting of the southeastern section of the Mathematical Association of America, April 16, 17, in Nashville, Tenn. The topics of his addresses were, *The Discovery of the Freedom to Inquire, and Abelian and Tauberian Theorems.* A dinner was given in honor of Dr. Carmichael. Over 250 mathematicians were invited. Seven southern states were represented. Twenty-eight mathematical papers were read during the meeting.

The twenty-second annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, April 1, 1937 with an afternoon session, a dinner, and an evening session. Professor J. H. Weaver, chairman of the Section, presided at these sessions.

The following papers were read:

1. *A Generalization of the Circles of Appolonius and Some Resulting Properties.* Professor J. H. Weaver, Ohio State University.
2. *Miscellanea on Diophantine Equations.* Professor I. A. Barnett and Dr. C. W. Mendel, University of Cincinnati.
3. *Geometric Proofs of Multiple Angle Formulas.* Professor Wayne Dancer, University of Toledo.
4. *A Note on Tensor Analysis.* Professor F. C. Jonah, Western Reserve University.
5. *Phythagorean Numbers.* Professor O. E. Brown, Case School of Applied Science.

6. *Solution of a Differential Equation of the First Order and First Degree in Terms of Infinite Series of Solutions of Linear Differential Equations.* Professor Jesse Pierce, Heidelberg College.
7. *Inconsistent Uses of Functional Notation.* Dr. C. C. Torrance, Case School of Applied Science.
8. *A Generalization of Newton's Identities.* Mr. H. Reingold, University of Cincinnati, introduced by Professor Barnett.
9. *Recent Developments in the Conjugate Endpoint Problem in the Calculus of Variations.* Professor George Saute, Western Reserve University.
10. *The Technique of Generalization.* Professor Henry Blumberg, Ohio State University.

Problem Department

Edited by
ROBERT C. YATES

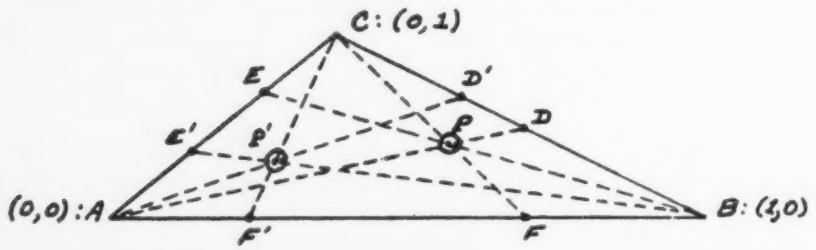
This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

No. 81. Proposed by *Walter B. Clarke*, San Jose, California.

Through a point P inside or outside the triangle ABC, draw a set of cevians cutting the sides (prolonged if necessary) at D, E, and F, respectively. Transpose the segments of each side obtaining points D', E', F', respectively. AD', BE', and CF' will be concurrent at P'. What is the locus of P so that PP' will be parallel to one of the sides of the triangle?

Solution by *Robert C. Yates*, University of Maryland.



Let $FB : BD : DC = \lambda : \mu : \nu = CE : EA : AF$. Using oblique reference axes, the triangle ABC may be taken with vertices at (0,0); (1,0); (0,1). The equations of the three cevians thus determined are:

$$(\lambda+\mu)y+\mu x=\mu$$

$$\nu y+(\lambda+\nu)x=\nu$$

$$\nu y-\mu x=0$$

which meet at the point P whose coordinates are:

$$(1) \qquad x=\nu/S_1; \quad y=\mu/S_1,$$

where $S_1=\lambda+\mu+\nu$.

Transposing (or interchanging)segments on each side, we have the ratios:

$$AF' : D'C : D'B=\lambda : \mu : \nu=E'A : CE' : F'B.$$

These ratios determine three other cevians whose equations are:

$$(\lambda+\mu)y+\lambda x=\lambda$$

$$\lambda y+(\lambda+\nu)x=\lambda$$

$$\mu y-\nu x=0.$$

These meet at the point P' whose coordinates are:

$$(2) \qquad x'=S_2/(\nu S_2); \quad y'=S_2/(\mu S_2)$$

where $S_2=\mu\nu+\lambda\nu+\lambda\mu$, $S_3=\lambda\mu\nu$.

If now we ask that PP' be parallel to a side, say AB, then the ordinates of P and P' must be equal. That is, from (1) and (2):

$$(3) \qquad \mu/S_1=S_3/(\mu S_2) \text{ or } (\mu^2-\lambda\nu)(\lambda+\nu)=0.$$

We have then two cases to consider:

1.° If $\nu=-\lambda$, we may eliminate ν in (1) to find:

$$y=1,$$

a line through C parallel to the side AB of the given triangle.

2°. If $\mu^2=\lambda\nu$, we may again eliminate ν in (1) to find:

$$1/x=\lambda^2/\mu^2+\lambda/\mu+1; \quad 1/y=\lambda/\mu+\mu/\lambda+1$$

or $1/x=t^2+t+1; \quad 1/y=t+1/t+1, \quad (t=\lambda/\mu),$

as the parametric equations of the locus of P. We have, on eliminating the parameter, t, the implicit equation:

$$\boxed{x^2+xy+y^2=x}$$

an ellipse inscribed to the parallelogram, $x=0$, $y=1/3$, $x=4/3$, $y=-1$, touching the sides AC and BC at A and B.

No. 126. Proposed by *Brother Northbert*, University of Portland, Portland, Oregon.

If the line L is the polar of the point P in respect to a family of circles, then the circles have a common radical axis which is the perpendicular bisector of the perpendicular line segment joining the pole with the polar.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

Let the pole P be $(k,0)$ and the polar L be $x=0$. The equation of the family of circles is then

$$(x-a)^2+y^2=c^2,$$

where $a^2-c^2=ak$.

Now any two circles of the family with radii c_1 and c_2, centers at $(a_1,0)$ and $(a_2,0)$, respectively, will have a radical axis:

$$2(a_1-a_2)\cdot x=(a_1^2-c_1^2)-(a_2^2-c_2^2)$$

$$=a_1k-a_2k.$$

That is, the common radical axis of the family is $x=k/2$.

No. 127. Proposed by *Richard A. Miller*, University of Iowa.

Using the notation, I for incenter, O for circumcenter, K for Nagel point (the intersection of the perpendiculars from the excenters to the sides), T for the centroid of the triangle formed by the excenters, I', I'', I'''; prove T, I, O, K are collinear and $IO : OT : TK = 3 : 1 : 2$.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

The external and internal bisectors of a triangle at any vertex are perpendicular. Thus I is the orthocenter of $I'I''I'''$. Furthermore, K is the circumcenter of $I'I''I'''$.*

The external and internal bisectors of a triangle at any vertex are perpendicular. Thus I is the orthocenter of $I'I''I'''$. Furthermore, K is the circumcenter of $I'I''I'''$.* The given triangle is the pedal of $I'I''I'''$ and its circumcenter is the nine-point center of $I'I''I'''$. It is well known that the orthocenter, the nine-point center, the centroid, and the circumcenter (I, O, T, K of $I'I''I'''$) are collinear and form the ratio:†

$$IO : OT : TK = 3 : 1 : 2$$

*See, for example, Altshiller-Court, *College Geometry*, p. 195.
†*Ibid.*, p. 165.

No. 135. Proposed by *Benedict C. Schwanda*, Colgate University.

Show that there exist two and only two palindromic pentagonal numbers between 1000 and 10000. A pentagonal number is of the form $n(3n-1)/2$, with n a positive integer. A palindromic number reads alike frontwards and backwards.

Solution by the Proposer.

Given: $P=abba$. Place $P=n(3n-1)/2$. It is seen that P lies between 1001 and 9999, whence $26 \leqq n \leqq 81$. Since P is divisible by 11, then n or $(3n-1)$ must be divisible by 11.

If $n=11\cdot k$, we must consider $n=33$, 44, 55, 66, and 77.

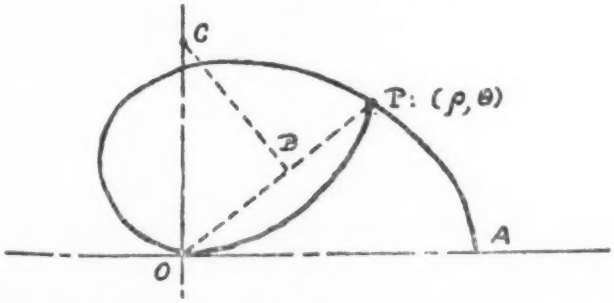
If $(3n-1)=11\cdot k$, we must consider $n=26$, 37, 48, 59, 70, and 81. On examining these values of n, the only pentagonal numbers which are palindromic result from $n=26$ and 44, and are 1001 and 2882.

Also solved by *W. C. Janes* and *C. W. Trigg*.

No. 136. Proposed by *J. S. Robberson*, Dallas, Texas.

Find the locus of the point P which moves so that the circular arc which joins P to a fixed point O is (1) always tangent to a fixed line through O and (2) always a fixed length a.

Solution by *E. H. Umberger*, University of Maryland.



Let the fixed line to which the circular arc is tangent be the polar axis and O the pole. The arc has its center at C. We take $OC=r$ and $OB=\rho/2$. Then, if $BOA=OCB=\theta$,

$$\rho/(2r)=\sin\theta.$$

Furthermore,

$$r\cdot(2\theta)=a.$$

Eliminating the parameter r, we have:

$$\rho \cdot \Theta = a \cdot \sin\Theta$$

as the locus of P.

Also solved by *W. C. Janes, W. B. Clarke, R. W. Clack,* and the proposer.

No. 137. Proposed by *Frank Morley,* Johns Hopkins University.

Semicircles are erected upon the given segments AB, BC, CA, of a line. Find the diameter of the circle which touches the three arcs.

Solution by *J. Rosenbaum,* Bloomfield, Conn.

Let P, Q, R be the mid-points of AB, BC, CA and let the lengths of AB and BC be $2a$ and $2b$. Also let the center of the fourth circle be O and its radius x.

Applying the Law of Cosines to each of the triangles, PQO and PRO, using in each case the common angle at P, we have:

(1) $$(b+x)^2-(a+x)^2-(a+b)^2=2(a+x)(a+b)\cdot\cos P,$$

and

(2) $$(a+b-x)^2-(a+x)^2-b^2=2(a+x)\cdot b\cdot\cos P.$$

On dividing these equations, there results a linear equation in x the root of which is

(3) $$x=ab(a+b)/[(a+b)^2-ab].$$

It is observed that this can be written as:

(4) $$\boxed{1/x=1/a+1/b-1/(a+b)}$$

On transposing, we have: *The sum of the curvatures of the largest and smallest of the four circles equals the corresponding sum of the other two circles.*

Solution by *G. F. Alrich,* University of Maryland.

Consider the circles:

(1) $$x^2+y^2=ax$$

(2) $$x^2+y^2=bx$$

and the train, T, of circles:

$$(x-a)(x-b)+(y-nc)^2=0, \quad (n=0,1,2,\cdots)$$
$$(c=a-b>0)$$

consisting of mutually tangent circles enveloped by the lines $x=a$ and $x=b$. The power of the origin with respect to the generic circle of the train is:

$$P_n=ab+n^2c^2=P_0+n^2c^2.$$

Between three consecutive P's there exists a linear relation:

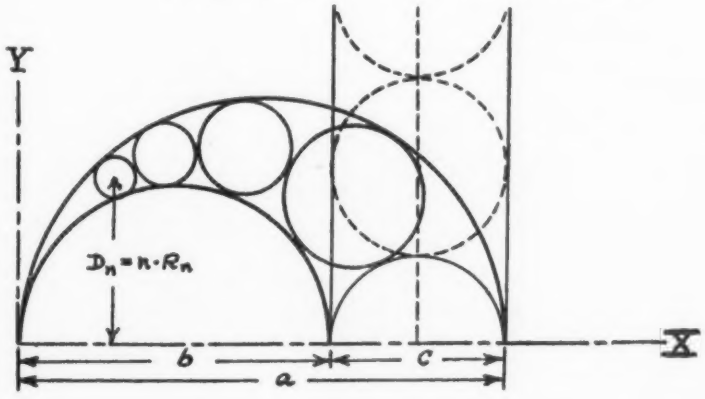
$$P_{n+1}+P_{n-1}=2\cdot(P_n+c^2). \tag{3}$$

Taking the orgin as a center of inversion and $P_0=ab$ as the square of the radius of inversion, the train T inverts into a train $\overline{T}$ of mutually tangent circles enveloped by the circles (1) and (2)*. If k_n is the curvature of the generic circle of $\overline{T}$, we have:

$$P_n : P_0=k_n : k_0$$

and equation (3) becomes:

$$(k_{n+1}+k_{n-1})/2=k_n+k_0\cdot c^2/P_0=k_n+2c/P_0 \tag{4}$$†



Since $c=a-b$ and $P_0=ab$, this expression may be reduced to the form:

$$(k_{n+1}+k_{n-1})/2=k_n+2/b-2/a$$

or

$$\boxed{(k_{n+1}+k_{n-1})/2=k_n+B-A} \tag{5}$$

*i. e., the circles into which the lines $x=a$, $x=b$ are inverted.—Ed.
†Since the circle for which $n=0$ is invariant under this inversion.—Ed.

where A and B are the curvatures of the circles (1) and (2), respectively. Thus, *the difference between the curvature of any circle of the train* $\overline{T}$ *and the average curvature of its two adjacent circles equals the difference of the curvature of the two enveloping circles.*

The solution of the problem proposed is found by making $n=0$ in equation (5). Since, by symmetry, $k_{-1}=k_1$, we have:

$$k_1=k_0+B-A,$$

or

$$\boxed{k_1/2+1/a=1/b+1/c.}$$

Editor's Note. Prompted by the method Alrich has employed, let us take a circle of radius $c/2$ and center at the point $(h,n\cdot c/2)$, n being any selected real number and $h=(a+b)/2$. (See figure). Its equation in complex variable is:

$$(x-h-inc/2)\cdot(\bar{x}-h+inc/2)=c^2/4$$

or

$$x\cdot\bar{x}-A\cdot\bar{x}-\overline{A}\cdot x=-\rho^2 \tag{1}$$

where $A=h+inc/2$; $\rho^2=A\cdot\overline{A}-c^2/4$; the bar, as usual, denoting a conjugate.

Inversion in a circle of radius K and center at the origin is performed by the transformation:

$$x\cdot\bar{y}=K^2$$

Inverting (1) by this, we have:

$$K^4/y\cdot\bar{y}-A\cdot K^2/y-\overline{A}\cdot K^2/\bar{y}=-\rho^2$$

a circle whose equation may be rewritten in the form:

$$(y-A\cdot K^2/\rho^2)(\bar{y}-\overline{A}\cdot K^2/\rho^2)=A\cdot\overline{A}\cdot K^4/\rho^4-K^4/\rho^2 \tag{2}*$$

$$=(K^4/\rho^4)(A\cdot\overline{A}-\rho^2)=K^4\cdot c^2/4\cdot\rho^4.$$

By inspection we find the center of this circle at the point:

$$A\cdot K^2/\rho^2=(h+inc/2)\cdot K^2/\rho^2, \tag{3}$$

*This is the generic circle of curvature k_n in the train $\overline{T}$ of Aldrich's solution.

and its radius:

$$R_n = K^2 \cdot c/2 \cdot \rho^2. \tag{4}$$

If we denote by D_n the distance from this center to the axis of reals, then, from (3) and (4), we have:

$$\boxed{D_n = n \cdot R_n} \tag{5}$$

This property seems remarkable indeed in the light of the following illustrations:

1°. If the lowest circle of the train $\overline{T}$ (or of T) be taken with its center on the axis of reals (the horizontal axis in the figure) and all the rest tangent to each other, then the distances D divided by their corresponding radii form the sequence of even numbers:

$$0, 2, 4, 6, 8, \cdot \cdot \cdot \cdot$$

2°. If the lowest circle of the train be taken tangent to the real axis, the distances D divided by their corresponding radii give the sequence of odd numbers:

$$1, 3, 5, 7, 9, \cdot \cdot \cdot \cdot$$

For other properties of the Shoemaker's Knife (the Arbelos), see the Fourth Book of Pappus's Mathematical Collection; Steiner's *Gesammelte Werke*, Vol. I, pp. 47-76; and an address by Mackay in the Proceedings of the Edinburgh Mathematical Society, Vol. III, p. 2.

Also solved by *F. A. Rickey, R. W. Clack, V. Higgins, C. W. Trigg, K. W. Crain,* and *J. S. Robberson.*

PROPOSALS

No. 144. Proposed by *R. A. Miller,* State University of Iowa.

Show that if the centroid of a triangle lies on the inscribed circle, then:

$$5(\sin^2 A + \sin^2 B + \sin^2 C) = 6(\sin A \sin B + \sin A \sin C + \sin B \sin C).$$

No. 145. Proposed by *E. H. Umberger,* University of Maryland.

Given the quadrilateral $ABCD$ with the points E, F dividing the opposite sides DC and AB, respectively, in the ratio $\lambda : \mu$, where $\lambda + \mu = 1$; $\lambda, \mu > 0$.

Prove: $$EF \leqq \lambda \cdot AD + \mu \cdot BC.$$

No. 146. Proposed by *A. Blake*, U. S. Coast and Geodetic Survey, D. C.

If two angle bisectors of a triangle are equal, the sides to which they are drawn are equal. Generalize this to the case of $n-$sectors, i. e., lines which divide the angles in the ratio $n : 1$ instead of $2 : 1$.

No. 147. Proposed by *A. Moessner*, Nurnberg—N, Germany.

What is the general solution of the diophantine system:

$$\left.\begin{aligned} 2x^5 &= a^4+b^4+c^4 \\ 2x^{10} &= a^8+b^8+c^8 \end{aligned}\right\} \ ?$$

No. 148. Proposed by *V. Thébault*, Le Mans, France.

With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken twice, form two numbers which are respectively the square and the cube of the same number.

No. 149. Proposed by *V. Thébault*, Le Mans, France.

Find a perfect square of ten digits of the form:

$$aabbccddee.$$

No. 150. Proposed by *M. H. Martin*, University of Maryland.

Show that $\cos\pi x \cdot \cos\pi y = \rho$, where $0<\rho<1$, and $-1/2 \leqq x,y \leqq +1/2$, is cut by a line in at most two points.

No. 151. Proposed by *A. C. Briggs*, Wilmington, Ohio.

A regular hexagon is composed of jointed rods, each of weight W. It is placed in a vertical plane with one rod resting on a table. Its right- and left-hand vertices are connected by a horizontal string of negligible mass. Find the tension in this string if a weight M is placed at the mid-point of the top rod.

No. 152. Proposed by *R. E. Gaines*, University of Richmond, Virginia.

If OA_1, OA_2, OA_3 be the semi-diameters drawn through the mid-points, M_1, M_2, M_3, of the sides of a triangle inscribed in an ellipse, and if the ratios $OM_1 : OA_1$, etc., be denoted by r_1, r_2, r_3, then

$$r_1^2+r_2^2+r_3^2 \pm 2r_1r_2r_3 = 1,$$

the double sign being plus or minus according as the center of the ellipse is inside or outside the triangle.

No. 153. Proposed by *R. E. Gaines*, University of Richmond, Virginia.

A variable line is tangent to $\rho = a(1+\cos\theta)$ at P and intersects the curve in two other points, Q and R. Prove that the locus of the midpoint of the chord QR is a limacon whose equation (with suitable change of origin) is $2\rho = a(1+4\cos\theta)$.

No. 154. Proposed by *W. V. Parker*, Louisiana State University.

Three isosceles triangles with base angles α, β, γ, are constructed on the sides a, b, c, of a triangle as bases. If $\alpha+\beta+\gamma = \pi/2$ and D, E, and F are the vertices of the isosceles triangles, prove that the angles of the triangle DEF are $\pi/2-\alpha$, $\pi/2-\beta$, $\pi/2-\gamma$.*

No. 155. Proposed by *Walter B. Clarke*, San Jose, California.

Given the incenter, circumcenter, and verbicenter of a triangle. Construct "in situ" the incircle, the circumcircle, and the nine-point of the circle. Can the triangle then be constructed?

No. 156. Proposed by *Walter B. Clarke*, San Jose, California.

Under what conditions will the line joining the incenter and the verbicenter of a scalene triangle be perpendicular to a side?

*For a special case of this theorem, see Problem No. 143, *N. M. M.*, Vol. XI, No. 7.

Reviews and Abstracts

Edited by
P. K. SMITH

Cours de Mathématiques Spéciales. Tome II. By H. Commissaire and G. Cagnac. Masson, Paris, 936. 443 pages.

Tome II contains fourteen chapters and forms Book III of the course. Function of a function, inverse functions, exponential functions, logarithms; relative magnitude of a^n, x^m, $\log_a x$, for x large; the number e, hyperbolic functions; series (both real and complex terms) and infinitesimals are studied in the first four chapters. In the next four chapters derivatives, the theorem of the mean, Taylor series (both limited and infinite), operations with series (addition, multiplication, division, differentiation, etc.), undetermined forms and tracing of curves given in the form $y=f(x)$ are taken up. Chapter nine is devoted to a discussion of the separation and location of the roots of equations. The remaining chapters cover curve tracing (parametric form, polar coordinates, algebraic curves), functions of several variables, differentials, surfaces (including ruled surfaces and developable surfaces), tangential equations of curves and surfaces, and the theory of envelopes (curves and surfaces).

As in Tome I the discussions are carefully made with a judicious balance of function theory, geometry and vector analysis. In the event that a theorem is stated without proof the student is not unduly frightened by any statement that the proof is too advanced. The mathematics major or the technical student who knows the scientific treatment of curve tracing as given in this volume would experience no difficulty in applications. It is refreshing to see a careful and well-illustrated discussion of this important topic. There are many typographical errors, some of which are noted in a short errata in Tome II; a more nearly complete list is given in Tome III.

Virginia Military Institute. WILLIAM E. BYRNE.

Plane Trigonometry. By Edward S. Allen. McGraw Hill Company, New York, 1936. 152 pages.

The author, in his preface, gives two main reasons for writing a new trigonometry:

1. To avoid the "logical defects" of defining trigonometric functions with the aid of rectangular axes.

2. To write a text specifically adapted to the McGraw-Hill Six-place tables.

The first chapter is introductory, dealing with vectors and angles. It is obvious that the author intends to avoid the use of rectangular axes when, on page 6, the discussion of quadrants requires a small diagram showing the location of quadrants I, II, III, and IV. Since this diagram amounts to a set of cartesian axes, this reviewer felt that the first aim of the author was partially defeated. At any rate this description of quadrants is probably as much a logical defect in his scheme as the one stated in the preface, "In a problem involving the functions of several angles, it (the use of axes) makes the awkward demand that each angle have its own pair of coordinate axes."

The second chapter is a very clear explanation of the six fundamental trigonometric functions and their values for special angles. A three-place table is given in this chapter. The third chapter gives the elementary identities. Chapter IV describes the use of the Six-place Tables, the solution of right triangles and some useful hints in the technique of computation. Chapter V, Functions of any Angle, is handled very cleverly without the use of axes, although the figures in pages 44, 45, 46 and 47, showing the right triangles in the position they would occupy if axes were present, seem to imply the presence of axes.

Chapter VI, which is called Combination of Angles, begins with a discussion of projections and vectors. The groundwork is laid for the derivation of sin $(x+y)$, using projections of vectors. The other related identities are derived in this chapter. Graphs of the inverse functions are given, and the proof of identities and the solution of trigonometric equations are described.

Chapter VII deals with the solution of the general triangle, while the last chapter treats the complex number system.

The number of exercises in this text is not large. In this respect it is in keeping with the general scheme of the book; brevity and conciseness are quite evident throughout.

Typography and format are excellent. Important formulas are printed so as to stand out clearly. The proofs of the law of cosines and the law of tangent are remarkably direct. Explanations in all chapters are easy to understand, and make the text useful from the student's standpoint.

Professor Allen's book deserves the attention of those contemplating a change in trigonometry text.

Santa Monica Junior College. L. J. ADAMS.

Men of Mathematics. By E. T. Bell. Simon & Schuster, New York, 1937. xxi+592 pages; 29 portraits. $5.00.

There exists today a strong tendency to pay homage and attention to the heroes of science, to humanize its essential creators, and this book is good evidence of the fact. A. Macfarlane (1916), H. W. Turnbull (1929), and G. Prasad (2 Vols., 1933-1934) have issued collections of mathematical biography, but this one is done on a much larger and more pretentious scale. A parade of the following mathematicians passes in rapid review: Zeno, Eudoxus, Archimedes, Descartes, Fermat, Pascal, Newton, Leibniz, the Bernoulli family, Euler, Lagrange, Laplace, Monge, Fourier, Poncelet, Gauss, Cauchy, Lobachevsky, Abel, Jacobi, Hamilton, Galois, Sylvester, Cayley, Weierstrass, Sonja Kovalevsky, Boole, Hermite, Kronecker, Riemann, Kummer, Dedekind, Poincaré, and G. Cantor. Of course much incidental information about other important mathematicians is included. From the above, it will readily be seen that the volume is devoted almost exclusively to the modern period; also the first three names are compresssed into one chapter. Gauss receives 52 pages, considerably more than the amount of space devoted to any other mathematician. The sketch of Gauss is based largely on Sartorius' monograph. Albrecht Dürer's *Melencolia* is reproduced as the frontispiece. The portrait illustrations, taken primarily from the David Eugene Smith Library, add much to the value of the volume.

Footnotes and bibliography are in general excluded and sources or authority for statements are usually not indicated. The author has purposely done this, since the aim of the book is popularization, or an appeal to the general reader. He specifically states that it is not intended to be a history of mathematics, or any section of it. He has based this colorful readable book on printed sources, as far as the present reviewer can ascertain. Most of them are obscure and not very accessible to the general reader. Dr. Bell is a seasoned, skilful writer with a fluent style; he writes with a realistic, curt, potent wit and stark, frank humor which does not stop short of vigorous, rollicking slang (regarded by some as "undignified"—but this is a matter of taste). Doubtless this will amuse many casual, rapid readers, but it also leads him to a certain type of exaggeration. Hence the historian will be largely interested in the *mode* of treatment, provided most of the facts are already familiar to him. Without attempting to do full justice to the skill of presentation within the limits of so brief a review, it may be said that the circumspect, careful reader will find a considerable number of examples of such "exaggeration". The author states

that an ordinary high school course in mathematics plus interest and an undistracted head is sufficient to understand everything presented.

Dr. Bell frequently recounts the charming legends, interesting traditions, and melodramatic fiction concerning the various mathematicians, but fortunately he usually discounts them properly by elucidating that they may be apocryphal anecdotes, rather than fact.

Occasionally slips occur, such as on p. 130, where we read: "Today, over three hundred years after his death, Leibniz's reputation as a mathematician is higher...." 1839 was not the only occasion on which Jacobi visited Gauss (p. 331). Herbart died in 1841, about five years before Riemann went to Göttingen (p. 491). The reader also finds certain errors of others presented as fact, such as Landau's statement that Dedekind was "the last pupil of Gauss" (p. 516). Also, Dedekind was professor in an "Institute of Technology", rather than a technical "high school" and this should not be thought inferior to a university, if we stop to consider the German system, (p. 518). He went there in 1862, probably glad to return to his home city after being in a foreign country (Zurich). The ties which grew with the years must have held him there; an alumnus often returns to his alma mater at a certain sacrifice, in order to teach there, and remains stuck there for the remainder of his life.

It is worth noting that no American was of the proper calibre to be included in this volume; J. J. Sylvester is the closest approach, and mention of this first sojourn in America is exceedingly brief. Any writer is always puzzled to know just what spelling to adopt for many foreign geographical and proper names, since often there is no standard.

Dr. Bell allows his imagination to play; conjecture, personal opinion, and speculation are abundant throughout these chapters as to what the historical development would have been if conditions had been different, or some mathematicians had lived longer, behaved differently, or if mortals were constituted otherwise.

To write such a work requires a huge amount of labor, but the author will probably be rewarded by finding a large circle of readers. Its laudable merit of popularizing, if this succeeds, will justify its publication as well as the toil required to produce it. The paper, printing, binding, etc., are all that one could desire. Misprints are extremely rare.

University of Illinois. G. WALDO DUNNINGTON.

Applications, the Better Half of Mathematics. (Abstract of a paper by Wilson L. Miser read before the Louisiana-Mississippi Section of the Mathematical Association of America, March 5, 1937.)

In all instruction, whether in high school or college, mathematics should be associated with everything that is seriously interesting to the student at the particular stage of his development and that can in anyway be brought in relation to mathematics, for as Felix Klein* says, "The living thing in mathematics, its most important stimulus, its effectiveness in all directions, depends entirely upon its applications, that is, upon the mutual relations between those purely logical things and all other domains."

In this paper the author contends (1) that the teacher can be sure that the student has really learned mathematics if he can apply it, (2) that it is through his ability to make applications that the student comprehends principles and feels the satisfying sense of his grasp of them, and (3) that the instructor can make applications a very effective aid in teaching the fundamental notions, principles, and methods of mathematics. To the student mathematics and its applications furnish an inexhaustible variety of exercises and problems which can be graduated precisely to the degree of attainment at any particular stage and yet be of interest and value.

In mathematical instruction the two principal aims are these: (1) the student should gain a sharpened understanding of the mathematical branch and a clear insight into its significance and (2) he should acquire the conviction that correct thinking based on true premises secures mastery over the outer world. To accomplish these aims the physical world must receive its share of attention.

The responsibility for improvement in teaching rests upon the the ability of teachers of elementary mathematics to hold securely to both aims in their teaching; upon the ability of teachers of science to develop their methods of instruction in such a way that mathematics applied to their subject becomes an effective and indispensable aid not only to themselves in teaching but also to the students in learning; and upon the ability of the students themselves to coordinate their study of mathematics and applied science in such a way that they may avail themselves of all the advantages of learning subjects together. Cooperation in aims and methods of teaching and coordination of program of studies are imperatively needed in simplifying and smoothing the teaching and learning processes for instructors and students.

*Felix Klein, Elementary Mathematics from an Advanced Standpoint, translated by E. R. Hedrick and C. A. Noble, p. 16.

Mathematics as a Subject Prescribed for Graduation from College. (Abstract of a paper by J. N. Michie read before the Texas Section of the Mathematical Association of America, April 24, 1937.)

This paper reviews the position of mathematics in relation to the requirements for the A. B. degree since the introduction of the elective system into American "higher" education, and points out the reason for its early inclusion as a required course, namely, that it was recognized as affording a mental discipline not provided by other studies. The opposition of present-day educators to this older attitude is briefly summarized, their disbelief in the concept of "mental discipline" and the "transfer" of learning skills, and their current tendency to meet what they call "individual differences" by altering degree requirements, often at the expense of academic standards is discussed. That none of their arguments for dropping mathematics from the list of required courses is entirely valid is asserted, and supported on the grounds that, whereas most studies are concerned with the matter of thought, mathematics is concerned with a method of thought, a method which is essential for logical thinking in any field.

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By

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This new text gives a comprehensive introduction to the theory of mechanics. An outstanding feature of the book is the excellent coordination the author has made between mathematical theory and physical proof. The author states that his "main object is to train the student to solve problems in mechanics" and has accordingly united a thorough study of the physical phenomena in the field of mechanics and the laws governing those phenomena with the best mathematical methods of the present day for solving the physical problems of mechanics. This is the first text thoroughly to coordinate the theory and physical principles of mechanics, and should give the student a better and more practical understanding of the mathematical theories presented. The teacher will find especially good material on Lagrange's Equations, Virtual Velocities, and d'Alembert's Principle, and on the fundamentals of the variational principles and the solution of Hamilton's Equations. *To be published in March or April.*

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THE SECOND DIMENSION OF ALGEBRA

The numbers of algebra correspond to the points of geometry. As an aggregate of geometrical points forms a geometrical space so an aggregation of numbers forms an algebraic space. Algebraic space, like geometrical space, may be measured in dimensions.

Algebraic space is absolute, geometrical space is relative, Zero is the absolute reference point of algebraic space. Any point, taken as origin, is the reference point of geometrical space.

Algebraic space is naturally polarized into positive and negative numbers. Geometric space is artificially polarized into positive and negative directions.

Algebraic space is naturally oriented in first, second and third dimensions. Geometric space is artificially oriented in x, y and z dimensions.

The numbers 1, 2, 3, and so on and the intervening fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ and their negatives -1, -2, -3, $-\frac{1}{2}$, $-\frac{1}{3}$, $-\frac{1}{4}$, correspond to the points of a straight line and constitute the first dimension of algebra.

The square root of minus one, with symbol i, is not a number of the first dimension of algebra. The multiples of i, that is, $i1$, $i2$, $i3$ and their intervening fractions, $i\frac{1}{2}$, $i\frac{1}{3}$, $i\frac{1}{4}$ and their opposites, $-i1$, $-i2$, $-i3$, $-i\frac{1}{2}$, $-i\frac{1}{3}$, $-i\frac{1}{4}$, constitute another dimension, the imaginary dimension of algebra.

Is this imaginary dimension the second dimension of algebra? No, logically, the imaginary numbers constitute the third, not the second, dimension of algebra.

When the numbers of algebra are developed in logical order two real dimensions precede the imaginary dimension.

The FIRST dimension of algebra is formed of the multiples and fractions of unity. This is the UNITARY dimension.

The SECOND dimension of algebra is formed of the multiples and fractions of the square root of positive unity. This is the HEMITROPIC dimension.

The THIRD dimension of algebra is formed of the multiples and fractions of the square root of negative unity. This is the IMAGINARY dimension.

This thesis is developed by Robert A. Philip in a monograph BIFOLIATE NUMBERS, Price one dollar, THE MONOGRAPHIC PRESS, 106 Washington St., Fairhaven, Mass.

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SHADOWS
OF NUMBERS

UPON the level land a fantastic shadow flits. It comes from nowhere; small at first, then larger, shapeless, then formed like humpbacked Punch. It springs forward, hestitates, shrinks, expands, now formed like a bull, but charging backwards, again it shrinks and vanishes.

In the sunlight, above the land, a square of paper floats. First, floating flat and edgewise to the sun it casts no shadow. The wind seizes it, crumples it, drives it forward, tosses it far upward, recrumples it, lets it fall, eddys it backward and again unfolds into a square, edgewise to the sun.

* * *

In mathematics there is a level space which is the habitat of the roots of algebraic equations. An algebraic equation of the nth degree has n roots, all complex numbers. All complex numbers dwell in one plane.

The plane of complex numbers is a self sufficient and self contained flatness. Shadows also dwell in such a self sufficient and self contained flatness.

He who gazes at shadows only can believe that the universe of shadows is all; beyond the shadows there is nothing; there is need of nothing; indeed, there can be nothing, for the universe of shadows is self contained. A self contained universe is an ultimate idea.

So the complex roots of an algebraic equation occupy a plane universe which seems to form an ultimate mathematical idea because it is self sufficient and self contained.

Every algebraic equation craves a solution. This craving is satisfied, or is apparently satisfied, by a superficial solution in complex numbers. This complex solution furnishes that illusion of thin, but perfect, self sufficiency which satisfies, throughly satisfies, a watcher of shadows.

Shadows crawl in the plane ot half truth while reality flies above through the space of whole truth. Reality flies with precision but shadows crawl in ambiguity. When lying shadow halts, who can tell whether flying reality halts or merely ascends or descends with undiminished speed?

* * *

Does a shadow desire to leave its habitat? Can a shadow spring upward out of flatness into space?

Consider a picture. Viewed with the naked eye this picture is flat, but the picture is a stereograph, which, viewed through a stereoscope does spring upward out of flatness.

There is a stereoscope for the eye of the body; there is also a stereoscope for the eye of the mind.

The roots of an algebraic equation are mathematical stereographs. Viewed directly the roots lie in the flat universe of the complex numbers; viewed through a mathematical steroscope the roots unfold into the boundless universe of the multifoliate numbers.

The superficial solution of an algebraic equation in complex numbers is but the shadow of a solid sclution in multifoliate numbers.

The complex roots of an algebraic equation are like the petals of a rose, pressed flat, the multifoliate roots of the equation are like the petals of a rose, full blown.

* * *

For those who desire stereographic views of the roots of algebraic equations, for the eye of the mind, we offer three monographs by Robert A. Philip.

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